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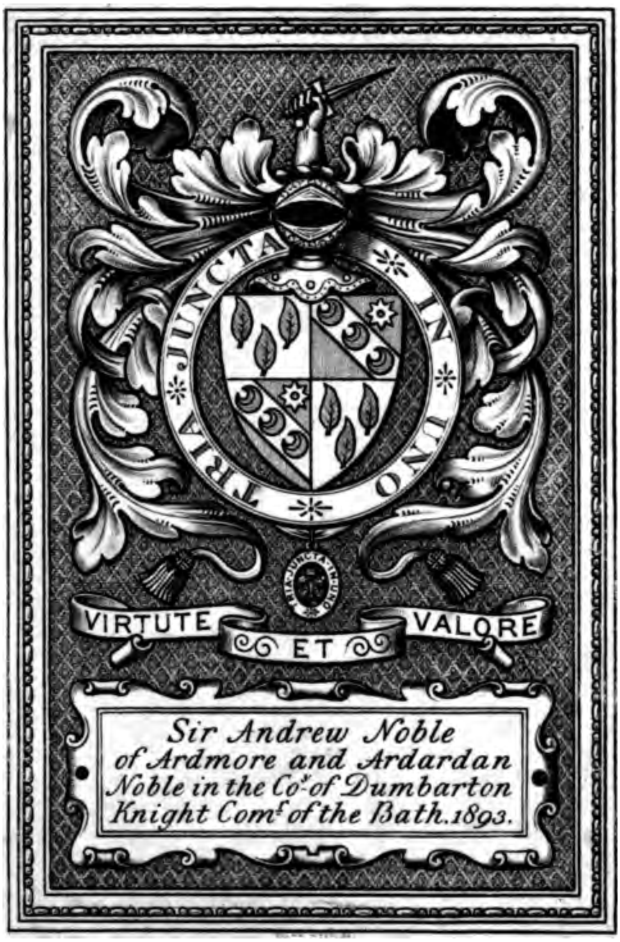

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STATICS

MINCHIN

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PUBLISHER TO THE UNIVERSITY OF OXFORD

LONDON, EDINBURGH, AND NEW YORK

A
TREATISE ON STATICS

WITH

APPLICATIONS TO PHYSICS

BY

GEORGE M. MINCHIN, M.A., F.R.S.

PROFESSOR OF APPLIED MATHEMATICS
IN THE ROYAL INDIAN ENGINEERING COLLEGE, COOPER'S HILL

VOL. I.

(EQUILIBRIUM OF COPLANAR FORCES)

FIFTH EDITION

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BY HORACE HART
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PREFACE

TO THE THIRD EDITION.

IT has been thought desirable to bring out the third edition of this work in two volumes, because experience proved that the previous edition contained more than was suitable to the wants of the great majority of students who reach the standard of Undergraduate Honours in Mathematics.

The reception of the work in the Universities at home and abroad has made me desirous of rendering it more deserving of the favour accorded to it by high-class students. Accordingly, I determined to reserve for the second volume the more advanced portions of the previous work (those dealing with Non-Coplanar Forces, Attractions, and the Theory of Strain and Stress), and, while greatly extending these portions, to introduce such fresh applications of the subject as would make the work really useful to those, for example, who aim at distinction in the Mathematical Tripos.

With regard to the first volume, little needs to be said. It is meant for those who attain the standard of Undergraduate Honours and Scholarships, but do not desire to compete for higher distinctions. Within this range it will, I think, be found tolerably complete. A re-arrangement of the order of treatment in the previous edition has been made. At an early stage in the student's reading I endeavour to make him familiar with graphic methods both in practice and in theory.

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To this end, attention has been directed to the solution, by graphic construction, of several classes of equations to which we are led in seeking for positions of equilibrium—equations, the accurate solution of which would be impossible, and the approximate solution of which by the ordinary analytical methods would be attended with great trouble. Experience has proved to me that this is a most valuable aid in producing in the mind of the student a knowledge of the nature of dynamical problems, and an interest which cannot be evoked by symbols and equations alone.

Indeed, it will be observed that graphic methods figure more largely in this edition, all through, than in the previous one—notably in the general discussion of Funicular Polygons in Chapter V. This is a branch of elementary Statics to which too little attention is paid; but it is both valuable and full of elegance.

The second volume opens with a long chapter on Non-Coplanar Forces, in which I have given an exposition of Sir Robert Ball's Theory of Screws, so far as it relates more particularly to the Statical branch of Dynamics. It will be observed that, while following Sir R. Ball's method of treatment very closely, I have departed from it in some instances. To prevent misconceptions, I may say that my reason for doing so is simply the fact that students of any branch of science derive great benefit by looking at it from several different points of view.

In this chapter the general conditions of equilibrium are illustrated by examples of the same character as those employed so largely in the chapters dealing with Coplanar Forces—my object being to avoid mere generalities in symbols.

The chapter on Astatic Equilibrium is founded on a paper which I published on the subject a few years ago. Now that students of the works of Hamilton, Tait, and Clerk Maxwell are so numerous in the Universities, no apology is necessary for the treatment of this subject by elementary Quaternions.

The part of this volume dealing with Attractions and the general theory of Potential in Electricity and Magnetism has been much enlarged; and it will be in its present condition, I hope, a valuable assistance to the student of the great and enduring works of Thomson and Tait, and Clerk Maxwell.

Many correspondents have been good enough to send me corrections of errors in the second edition. My obligations on this account are great to Professor Everett and Professor Schuster, whose corrections must have required care and trouble. Some American correspondents, also, have kindly sent corrections and suggestions; and among them I must chiefly thank Mr. F. Franklin of the Johns Hopkins University.

The proof sheets have all been revised by my friend and colleague Mr. W. G. Gregory, B.A., whose attainments in Physics, both practical and theoretical, rendered his criticisms of the utmost value.

GEORGE M. MINCHIN.

R. I. E. College, Cooper's Hill,
November, 1884.

PREFACE

TO THE FOURTH EDITION.

THE alterations and additions which chiefly characterise this edition may be briefly summarised. Chapter V, on the reduction of Coplanar Forces, has been rearranged and, to a considerable extent, rewritten, in such a way that the illustrations of particular principles follow more closely the expositions of these principles than in the previous edition; and an Article has been specially devoted to the *Application of Statics to Geometry.*

The deduction of certain well-known results in Geometry by means of the methods of another branch of Science cannot fail to be instructive to the student; and it must be interesting to see how Ptolemy's Theorem follows from the Parallelogram of Forces, and how the famous Theorem of Pascal is a simple result of a fundamental property of Funicular Polygons. This statical deduction of Pascal's Theorem is here, I believe, published for the first time; and a somewhat similar method should, apparently, be capable of deducing the allied Theorem of Brianchon; but, so far, I have not been able to find a statical proof of this latter.

The method of calculating the reactions at the joints in systems of bars has been more fully explained, and has been more largely exemplified by problems most of which are taken from, or suggested by, the papers set in the Mathematical Tripos.

The Chapter on the equilibrium of Strings has been rearranged in such a way that the discussion of Catenaries (under the action of gravity) precedes the general treatment of the subject. No doubt the student will find this a simplification.

In the revision of most of the proof sheets I have had the benefit of the care and criticism of my colleague Professor Stocker; while Mr. C. F. Findlay, late Fellow of Trinity Hall, assisted all through and not only freely and effectively criticised, but actually took the trouble of testing the results of nearly all the questions left for solution.

To both of these gentlemen I beg to express my obligations.

GEORGE M. MINCHIN.

June, 1890.

PREFACE

TO THE FIFTH EDITION.

In this edition I have added, in an Appendix, a statical proof of Brianchon's Theorem, for which I am indebted to Mr. E. M. Langley, and also a collection of examples, largely numerical, on the general theory of the reduction of a system of co-planar forces. A special section of these examples is devoted to problems on Catenaries under various conditions.

Such numerical examples I have always found of the utmost use in all branches of Mathematical Physics: indeed, without them, I hold that teaching must always be ineffective. It will also be observed that I have emphasised my belief in the practical value of graphic solutions.

GEORGE M. MINCHIN.

September, 1896.

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STATICS.

CHAPTER I.

THE COMPOSITION AND RESOLUTION OF FORCES ACTING IN ONE PLANE AT A POINT.

1. **Definition of Force.** *Force is an action exerted upon a body in order to change its state either of rest or of moving uniformly forward in a right line.*

This is the definition of Force given by Newton (Principia, Book I, Def. IV).

2. **Divisions of the Science.** The Science which treats of the action of Force on bodies is called *Dynamics*. Of this science there are two branches: the first treats of the laws to which forces are subject when they keep bodies at rest, and this branch is called *Statics*; the second treats of the laws to which the motions of bodies are subject when these motions are produced by given forces, and this branch is called *Kinetics*.

3. **Matter.** Matter is something which exists in space, and attests its presence by such observed qualities as extension, resistance, and impenetrability.

A limited portion of matter is called a *Body*, and the quantity of matter contained in a body is called its *Mass*. A very small portion of matter is called a *Particle*.

4. **Velocity.** Suppose a point to move along a right line in such a way that it always takes the same time, t, to travel over the same length, s, of the line, at whatever points of the line the extremities of this length are situated. Then we readily say that the point's 'rate of moving' is the same all through, and this rate we measure by the quotient $\frac{s}{t}$. The rate of moving we call the *velocity* of the moving point. But if the time of moving over the length s is not the same all through—i.e., if it depends

on the points of the line between which it is measured—the velocity, or rate of moving, is clearly not uniform. Nevertheless we recognise the fact that at each of its positions the moving point has a particular rate of going. How is this rate to be estimated? Like all rates, it must be measured by a differential coefficient. Thus, if P and Q are two extremely close positions, and if O is any fixed point on the line of motion, the distance between O and P being called s and the distance OQ being called $s + \Delta s$, and if the point has taken the infinitesimal time Δt to get from P to Q, we shall be very near the truth in assuming that its rate of moving has remained uniform in the passage from P to Q, and the velocity in this interval will, as above, be the quotient $\frac{\Delta s}{\Delta t}$. The smaller the interval PQ (and therefore the smaller Δs and Δt) the more nearly true is the assumption of uniformity of the rate of moving from P to Q. Hence if we could find the value of the ratio $\frac{\Delta s}{\Delta t}$ when both Δs and Δt are indefinitely diminished, we should have the exact rate of moving at P. But the limit of this ratio is the differential coefficient $\frac{ds}{dt}$, which is easily found by the rules of the Differential Calculus.

We have thus not only a conception of different rates of moving, but also a method of estimating these rates numerically at different points of the path.

5. **Criterion of the action of Force.** Instead of the motion of a mere mathematical point, let us consider the motion of a material particle. How can we tell whether this moving particle is acted on by *force* or not? The answer is—unless the particle is completely at rest, or, failing this, *moving with a uniform velocity in a right line*, it is acted on by some force. Observe the two distinct characteristics which must be possessed by the motion of a particle which is not acted on by force—the velocity must be constant in magnitude and the path must be a right line.

6. **Measure of Force.** Suppose a particle to move along a right line in such a way that in any interval of time, t, there is the same addition made to its velocity, between whatever epochs of time the interval t is reckoned. Then the velocity is

obviously increased at the same rate at every point of the path, and the particle is said to be continuously acted on by a *uniform force* in the line of motion. The time-rate at which this increase of velocity takes place is taken as the measure of the force acting on the particle; that is, if the same particle moves along a right line in such a way that its velocity is increased at a constant rate which is double the previous rate, it will be continuously acted upon in the second motion by a force which is double the previous force.

If the time-rate of increase (or in other words, the *acceleration*) of the particle's motion is not uniform, the force acting on it is not uniform, and its magnitude at any point of the particle's path is estimated by the rate of increase of the velocity of the particle at this point.

Since the velocity of one and the same particle is capable of having all possible rates of increase, all forces may be compared with each other by means of their effects on a single particle.

7. **Ways in which Force is produced.** One of the simplest ways in which a force can be made to act on a particle consists in attaching a string to the particle and pulling this string so as to cause the particle to move. If no other force acts on the particle, and if the string is always pulled in the same right line, the particle will continue to move in this right line; and the *rate*, per unit of time, at which its velocity is being increased at any point of its path is a measure of the magnitude of the force with which the string pulls it; so that if for any finite time we observed its velocity to remain constant, we should know that during this time the string ceased to be pulled, and that no force acted on the particle in this particular interval.

There are other ways in which forces act on particles, but the manner in which they act is not in every case known to us. For example, if the particle consists of a small piece of soft iron and we hold it near the pole of a magnet we shall see it rushing with continually increased velocity towards the magnet, and it is therefore, by definition, acted on by some force towards the magnet. This force can be measured, as before, at every point of the particle's path by the rate, per unit of time, at which it produces an increase of velocity in the particle; nevertheless it is quite uncertain how this force is produced—whether it is an action at a distance or a stress in some intervening medium.

But whatever its cause may be, we can measure it numerically by its effect—viz., time-rate of increase of velocity produced in a material particle.

Again, since the velocities of planets towards the sun and of meteoric stones towards the earth are perpetually accelerated, the planets are acted upon by forces towards the sun, and the meteors by forces towards the earth. These forces are called *forces of attraction*; but the nature or precise mode of operation of this attraction is a matter on which no certainty exists.

8. **Linear representation of Forces.** Consider a single material particle. Every velocity which it can have possesses three characteristics—it must have a certain numerical magnitude, it must take place in a certain right line, and it must take place in a certain *sense* (from right to left or from left to right) along this line; or, in other words, it must have *magnitude, line of occurrence,* and *sense.*

Now every velocity can be regarded as produced in the particle by the uniform action of a force for a definite time. Hence forces are also characterised by *magnitude, line of action,* and *sense.*

Two forces acting on a particle are therefore compared by specifying the two lines and senses in which they would cause it to move if each acted separately, and also the magnitudes of the velocities which they would thus generate in it if they both acted for the same time on it.

Hence any force may be completely represented by a right line drawn in the direction and sense in which it would cause a material particle to move, the length of this line representing, on any scale, the rate per unit of time at which the force would generate velocity in the particle. And all other forces may be compared with this force as to magnitude, direction, and sense by drawing right lines in the several directions in which they would produce motion, and taking the lengths of these lines to represent, on the same scale as before, the rates at which they would severally generate velocity in *one and the same particle.*

Forces may also be compared with each other by means of their effects on *different particles.* For, let n perfectly equal particles of the same material be placed side by side in a row (Fig. 1), and let each of them be acted upon uniformly for the same time by a force which at the end of this time generates the same velocity, *f*, in each of them. Now if instead of being *n* separate particles they were all glued together so as to form a body of *n* times the mass of each particle, and if each of them is still acted on by the same force as before, this body will, at the end of the time considered, have the same velocity as each separate particle had, and will be acted upon by *n* times the force which generated this velocity in the particle. Comparing a single particle, then, with the body whose mass is *n* times the mass of this particle, we see that to generate the same velocity in two bodies by forces acting on them for the same time, the magnitudes of the forces must be proportional to the masses to which they are applied.

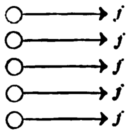

Fig. 1.

And hence, generally, if we define *momentum* as the product of mass and velocity—

The magnitude of a force is proportional to the rate per unit of time at which it generates momentum.

The greater the mass on which the force acts, the less the rate at which it increases the velocity of this mass; and the less the mass, the greater the rate of increase of velocity; the product of the two being always the same for the same force, *whatever be the masses to which it is applied.*

So that if P is a force which generates velocity at the rate $\frac{dv}{dt}$ in a body of mass m, and if P' is a force which generates velocity at the rate $\frac{dv'}{dt}$ (per unit of time) in a body of mass m', we have

$$\frac{P}{P'} = \frac{\frac{d}{dt}(mv)}{\frac{d}{dt}(m'v')}.$$

9. The C. G. S. system. Since the magnitude of a force is estimated by the time-rate at which it generates momentum, and since velocity involves length and time, we see that three distinct things are involved in the measure of force—viz., *length*, *mass*, and *time*. The questions then arise, what shall we take for the unit of length, what for the unit of mass, and what for the unit of time? For the purposes of calculation chiefly in Electricity and Magnetism, the system now adopted everywhere is one in which the *centimetre* is the unit of length; the mass of water at its temperature of maximum density which would just fill a cubic centimetre is taken as the unit mass and is called a *gramme*; and one mean solar second is taken as the unit of time. This system of units is called the "centimetre-gramme-second" system, or, more briefly, the C. G. S. system.

Hence a unit velocity is a velocity of one centimetre per second, and a force which, continuously acting on a gramme mass, increases its velocity by a velocity of one centimetre per second at the end of every second is the unit force. This unit force is called a *dyne*. Roughly, its magnitude is the $\frac{1}{981}$th part of the *weight* of the gramme mass in London.

For the ordinary purposes of commerce, force magnitude is often expressed in *kilogrammes* weight—a kilogramme being 1000 grammes. In England, where, unfortunately, a complicated and most absurd system of weights still prevails, force magnitude would in similar circumstances be expressed in *pounds* weight.

It must be carefully observed that the *weight* of a gramme mass is not a definite thing, because it is different at different places on the earth, being greater in high latitudes than in low; but the gramme mass itself—i.e., the quantity of matter called a gramme—is the same everywhere, whether on the earth or in any part of the universe.

10. **Equality of Two Masses.** We know by experience that an elastic string or a metallic spring exerts force when it is stretched beyond its natural length; and we can easily suppose that whenever the string or spring is stretched to a certain extent it will exert the same force. Moreover, the magnitude of this force could be expressed in dynes, by measuring the number of centimetres per second added every second to the velocity of our gramme mass of water (converted, for convenience, into ice) while the string or spring is attached to the mass and pulled at the given constant stretch; or, what comes to the same thing, by measuring the number of centimetres described by this mass at the end of any number of seconds under the influence of the constant pull exerted in a right line by the string or spring. The student will subsequently learn that, if we double this number of centimetres and divide by the square of the number of seconds during which the force has been applied, we shall obtain the number of dynes contained in the force. And we can imagine the stretch so graduated as to enable us to measure any number of dynes. Thus a force of any magnitude may con-

ceivably be measured by means of its effect on our standard gramme mass of water; and this very measurement will enable us to work with a body other than water—say platinum—by enabling us to define what we mean by a gramme mass of the new body. How do we know when we have two equal masses, one of water and the other of platinum? Indeed, before we answer this question, we must observe that there may be no *real* equality between such substances possible at all; any equality between them may be only a *conventional* equality. If all apparently different kinds of matter could be ultimately resolved into one simple substance—be it hydrogen or anything else—then a *real* equality of quantity of matter is possible between water and platinum, and conceivably either of these substances could be actually converted into the other. But if there is no *one* substratum at the basis of all bodies, it is impossible that any other than a *conventional* equality can exist between them.

The convention which is adopted for defining equal masses of two different kinds of bodies is this—

Two masses, one of a substance A and the other of a substance B, are defined to be equal when the same force produces exactly the same effect on both—for example, makes them both move over the same length in the same time, or generates velocity at the same rate in both.

11. Composition of Velocities. We propose to show how a particle may be moving with two velocities in two different directions at the same time. Let a board be placed on a horizontal table; let a rectilinear groove, *OA* (Fig. 2), be cut in this board, and let a particle be placed at *O* in the groove. Suppose, for definiteness, that the unit of time is one second. Let the particle be moved along the groove with a uniform velocity represented by *OA*, and at the same time let the board (i. e. every point in the board) be moved along a groove cut in the table with a uniform velocity represented in magnitude and direction by *OB*. Over what point in the table will the particle be found at the end of one second? Before the motions begin, complete the parallelogram *OACB*.

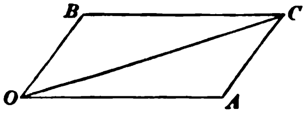

Fig. 2.

At the end of a second the particle must be found in the groove at the point A, and also at the end of the same time the point A of the groove must be found at the point of the table vertically under C. Hence this latter point is the position of the particle referred to the table at the end of a second.

Let the foot of a perpendicular dropped from the particle on the table be called *the position of the particle referred to the table.* How do we know that the position of the particle referred to the table has described the right line OC (or rather a line in the table vertically under OC)? In this way—if we demanded the position of the particle referred to the table at the end of any fraction or multiple of a second, we should find that the distance which it has travelled along OA is to the distance which the groove has travelled in the direction OB as OA is to AC, and therefore the positions of the particle referred to the table trace out a right line vertically under OC.

Consequently the two simultaneous velocities OA and OB which were impressed on the particle have combined to give it a single velocity represented in magnitude and direction by OC.

The velocity OC is called the *resultant* of the velocities OA and OB, and these latter are called *components* of the velocity OC. Hence we arrive at the proposition which is the foundation of Dynamics:—

If a point, O, move with two coexistent velocities represented in magnitudes, directions, and senses by two right lines, OA and OB, it will have a resultant velocity represented in magnitude, direction, and sense by the diagonal, drawn through O, of the parallelogram determined by the lines OA and OB.

This proposition is called by the name of *The Parallelogram Velocities.*

12. **Composition of Forces.** From the Parallelogram of Velocities, the *Parallelogram of Forces* follows at once. Since two simultaneous velocities, OA and OB, of a particle result in a single velocity, OC, and since these three velocities may be supposed to be produced by the separate action of three forces all acting for the same time, it follows that the effect produced on a particle by the combined action, for the same time, of two forces may be produced by the action, for the same time, of a single force which is therefore called the *resultant* of the other two forces.

And these forces will be represented in magnitudes, lines of action and senses by the lines OA, OB, and OC (Art. 8); hence—

If two forces be represented in magnitudes, lines of action, and senses by two right lines OA and OB, their resultant is represented in magnitude, line of action, and sense by the diagonal, OC, of the parallelogram OACB determined by these lines.

This is the proposition of the Parallelogram of Forces.

Cor. 1. The resultant of two forces acting along the same right line and in the same sense is equal to their sum; and if they act in different senses, the resultant is equal to their difference.

Cor. 2. The resultant of two forces acting at a point divides the angle between their lines of action into two parts whose sines are inversely proportional to the magnitudes of the forces.

13. **Equilibrium of three Forces.** In Fig. 2 produce CO through O to C' so that $CO = OC'$. Now imagine that, when the particle is started along the groove and the board along the table, the table itself is moved in a groove cut in the floor in the direction OC' with a velocity represented by OC'. In this case it is evident that the position of the particle with reference to the floor is fixed; that is, the particle is at rest with regard to fixed space (the floor being supposed fixed).

Consequently if three forces represented by the lines OA, OB, and OC' act together on the particle, no motion will ensue. In this case *each force is equal and opposite to the resultant of the other two*; for it is obvious that OA is equal and opposite to the diagonal, through O, of the parallelogram determined by OB and OC'; and that OB is equal and opposite to the diagonal of the parallelogram determined by OA and OC'.

14. **Statical point of view.** The primary conception of *force* is that of a cause of motion in a body or in a material particle, and the magnitude of any force is estimated by the time-rate at which it generates momentum (Art. 8). Nevertheless in Statics it is only the *tendency* which forces have to produce motion that is considered. Forces in this branch of Dynamics are considered as acting in such ways as to counteract each other's tendency to produce motion, or as producing a state of equilibrium in the bodies to which they are applied; but the magnitude of each force is estimated none the less with reference to the amount of momentum which it would *actually* generate if it were completely unfettered by the action of other forces.

In common usage Forces are often expressed as multiples of the *weight* of some standard body arbitrarily chosen. Thus, a force is said to be a force of 10 kilogrammes' weight if it is *just* capable of lifting vertically a body whose weight is equal to that of the mass of water which at a temperature of about 4°C. fills a volume of 10 cubic decimetres. But even here the Newtonian definition of force, as a cause of change of motion, is not discarded but merely kept in the background. For the weight which is called a kilogramme weight is merely a force which generates velocity at a certain rate in a body of certain mass; and the vertical force which is *just* able to raise a body from the ground is a force which could generate velocity in the body at the same rate as its weight and in the opposite sense. For practical purposes this measurement of forces as multiples of a weight is used by engineers and others; but, as has been already said in the branch of Dynamics which treats of Electricity and Magnetism, a different measure of force is resorted to—viz., a measure which is one and the same all over the earth, and indeed all through the universe. The mass of a body is something which cannot conceivably change, whether the body is taken to different parts of the earth or to different parts of the universe; and the force which, acting uniformly on this mass for a certain time, will at the end of this time have caused it to move with a certain velocity, must be one and the same wherever the experiment is tried. Hence the C. G. S. units are called *absolute units.*

15. **Absolute Measure and Gravitation Measure.** Were it not that the Earth is so nearly a spherical body whose strata of constant density are also nearly spheres having a common centre, it would probably not occur to men to measure forces as multiples of the *weight* of some definite standard quantity of matter—as, for example, a *pound* or a *kilogramme* of water—for the simple reason that the *weight* of this pound or kilogramme might be very greatly different at one place on the surface from its value at another place; for the *weight,* at any given place, of a given mass (pound or kilogramme of water) is simply the force with which the Earth attracts it at the place. Now if the Earth were an extremely flat ellipsoid of ununiform density, instead of being nearly the spherical body above described, we might imagine two steel bars of identical constitution, length, and cross section sent for examination of strength to two ob-

servers, one at the equator and the other near the pole; and the reports of these two observers (if both used gravitation measure) would be very different indeed. The observer at the equator might say that the bar broke under a pull equal to the weight of 10,000 kilogrammes, while the other observer might report the breaking force as the weight of 5,000 kilogrammes—the reason being that a mass of a kilogramme is twice as heavy near the pole as it is near the equator. Indeed, under such circumstances, the comparison of forces at different places on the Earth's surface would require such troublesome corrections for latitude, longitude, and other things, that the measuring of forces as multiples of the *weight* of some standard quantity of matter would be practically impossible, and the great distinction between the *mass* of a body and its *weight* would be clearly emphasised in the minds of men, instead of being (as it is at present) almost wholly lost in the case of the great majority of them.

With this gravitation method of measuring force let us contrast the *absolute* method of measurement. At all places the quantity of matter contained in a cubic centimetre filled with distilled water at its maximum density is the same, and the force which will produce a definite velocity in this mass by constantly acting on it for a definite time (suppose 1 c.m. per second every second) is the same; so that if the two observers supposed above reported the breaking force of the bars in *dynes*, or in *megadynes* (a megadyne $= 10^6$ dynes), their numbers would be absolutely the same.

Of course, as the Earth is not strictly the spherical body before described, the weight of a pound or a kilogramme of matter is not exactly the same at all places, but is less near the equator than it is near the poles. The amount of variation, however, is so small as to be practically negligible for most purposes. Nevertheless the distinction between *weight* and *mass* is one which the student should learn to make even thus early. It happens that the various Planets of our Solar System are all nearly spherical bodies, so that at all points on the surface of any one of them the weight of a given mass is nearly the same—though the magnitude of this force varies from Planet to Planet—and it is found from Astronomical calculations that the following numbers* represent, approximately, the weights of the same mass (suppose 1 gramme) on the surfaces of the following bodies:

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* Arago's *Popular Astronomy*, vol. ii. p. 415.

On the surface of the Sun	28·30
,, ,, ,, Mercury	·51
,, ,, ,, Venus	·91
,, ,, ,, the Earth	1·00
,, ,, ,, the Moon	·16
,, ,, ,, Mars	·50
,, ,, ,, Jupiter	2·45
,, ,, ,, Saturn	1·09
,, ,, ,, Uranus	1·05
,, ,, ,, Neptune	1·10

Hence if we imagine the same metallic spiral spring with a pointer and a dial plate so graduated into equal parts that when a mass of, say, 1 kilogramme is suspended from the end of the spring the pointer stands at the number 1 on the surface of the Earth, it would stand at these various numbers on the surfaces of the corresponding bodies of the System.

It is usual with engineers to speak of a force of so many *pounds*, or so many *kilogrammes*, or so many *tons*, instead of so many *pounds' weight*, or so many *kilogrammes' weight*, or so many *tons' weight*. This mode of speaking, however, is not one that will be adopted in this treatise, because it is considered desirable to restrict the terms *pound*, *ton*, *kilogramme*, *gramme*, &c. to *quantities of matter*, and not to employ them to denote the *weights* of such quantities of matter—the weights and the quantities of matter being obviously not the same things, as just explained. Practically, however, no harm can arise from the stereotyped mode of speaking.

16. **Force must act upon matter.** Although the Newtonian definition and measure of force render it clear that whenever force acts it must act on something material, it is not impossible that beginners may lose sight of this fact and suppose that a force could, for example, act on a mathematical *point*. We may without error speak of forces as acting *at* a point, but not *on* it, if their lines of action pass through the point. Thus in Fig. 2 two forces acting along the lines OA and OB may be spoken of as two forces acting *at* the point O; but their action would be physically impossible unless it took place on some material body, such as a particle placed at O. *Whenever force is exerted there must be matter,* both acting and acted upon.

17. **Proper Representation of Forces.** In representing the

resultant of two forces which act together at a point, O, the student should draw two lines marked with arrow-heads representing the two forces acting *from* the point. Thus, if of the two forces P and Q, one, P, is represented as acting *from* O, and the other *towards* O, we must produce the line QO to Q', so that $OQ' = OQ$; completing, then, the parallelogram $OPRQ'$, its diagonal, OR, will represent in magnitude and direction the resultant of P and Q. The marking of lines representing forces with arrow-heads will serve to exhibit the *senses* of the forces in every case.

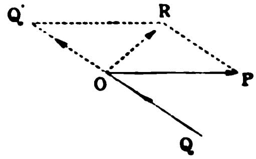

Fig. 3.

Again, since *force* magnitude and *linear* magnitude are two things of essentially different kinds, the magnitude of a force can never be *equal* to the length of a line, but it can be *represented* by such a length on a conventional *scale of representation*. Thus, if we are dealing with forces of 6 pounds' weight, 9 pounds' weight, and 21 pounds' weight, we may agree to represent 1 pound weight by a line 1 inch long, in which case the above forces would require lines 6, 9, and 21 inches in length to represent them respectively. The paper on which we draw may, however, be of insufficient size to allow conveniently of our drawing such a line as the last, in which case we might adopt a different and more convenient scale of representation—say a scale on which 1 inch of length represents a force of 3 pounds' weight, and then the lines required for the above forces would be only 2, 3, and 7 inches long, respectively. Thus the scale of representation is entirely dependent on convenience. If the forces are very large, 1 inch must be taken to represent a force of a great many pounds' weight; while if the forces are of very small magnitudes, a force of 1 pound weight may be represented by a line several inches long—all errors of calculation founded on the drawing being evidently much less with a scale of the latter kind than with one of the former.

18. **Resolution of Forces.** Having proved the principle of the Composition of Forces, the principle of the Resolution of Forces at once follows. If two forces, P and Q, are equivalent to a single force $OO' = R$ (Fig. 4), it is evident that the single force R acting along OO' can be replaced by the two

forces P and Q, represented in magnitudes and directions by two adjacent sides of a parallelogram of which OO' is the diagonal. Since an infinite number of parallelograms of each of which OO' is the diagonal can be constructed, the force R can be resolved in an infinite number of ways into two other forces. These forces are called *components* of R.

19. **Theorem.** It being given that the direction of the resultant of every two forces is that of the diagonal of their parallelogram, its magnitude must be represented by this diagonal; and conversely.

Let it be granted that the resultant of P and Q acts in the diagonal, OO' (Fig. 4), of the parallelogram determined by P and Q. Measure backwards through O a line, OR, the length of which represents the magnitude, R, of the resultant. A system of forces acting at O, represented in magnitudes and directions by P, Q, and R, will evidently be in equilibrium. Each force is, therefore, equal and opposite to the resultant of the other two. If, then, we consider P as equal and opposite to the resultant of Q and R, OP', the production of OP, must be the diagonal of the parallelogram determined by Q and R. Now, since $OQP'R$ is a parallelogram, $OR = P'Q$; and since $OP'QO'$ is a parallelogram, $P'Q = OO'$; therefore $OR = OO'$.—Q.E.D.

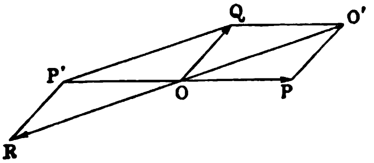

Fig. 4.

Again, for the converse proposition, let it be granted that $OR = OO'$, while OO' and OR are not necessarily in one right line; and let OP' be diagonal of the parallelogram, $OQP'R$, determined by OQ and OR; then OP is equal in magnitude to OP', since the resultant of Q and R has a magnitude equal to OP'.

Comparing the triangles OQO' and OQP', we have $OO' = QP'$, $QO' = OP'$, and OQ common to both; therefore the angle QOO' = the angle OQP', therefore QP' is parallel to OO'; but QP' is also parallel to OR, therefore OR and OO' are in one right line. Therefore, &c.—Q.E.D.

20. **Relations between three Forces in Equilibrium.** When three forces maintain a particle in equilibrium, each force is equal in magnitude to the resultant of the other two, and acts in the sense exactly opposite to this resultant. Thus, in Fig. 4,

each of the lines, OP, OQ, and OR, which represent in magnitude and direction the forces P, Q, R, is equal and opposite to the diagonal of the parallelogram determined by the two remaining lines.

This enables us to express the relative magnitudes of three forces in equilibrium by means of the three angles between them. For (Fig. 4) the forces P, Q, R are equal in magnitude to the lines OP, PO', $O'O$, respectively. Now, since the sides of a plane triangle are to each other as the sines of the opposite angles, we have

$$OP : PO' : O'O = \sin PO'O : \sin O'OP : \sin OPO'.$$

Denote by $\widehat{PQ}$, $\widehat{QR}$, $\widehat{RP}$, the angles between the directions of the forces P and Q, Q and R, R and P, respectively. Then, evidently,

$$\sin PO'O = \sin QOO' = \sin QOR = \sin \widehat{QR};$$

$$\sin O'OP = \sin ROP = \sin \widehat{RP}; \ \sin OPO' = \sin POQ = \sin \widehat{PQ}.$$

Hence we have the fundamental relations

$$P : Q : R = \sin \widehat{QR} : \sin \widehat{RP} : \sin \widehat{PQ}.$$

It may, perhaps, assist the beginner to mark the angle opposite to each force by the corresponding small letter (Fig. 5); and then the ratios of the forces may easily be remembered in the form

$$P : Q : R = \sin p : \sin q : \sin r. \qquad (a)$$

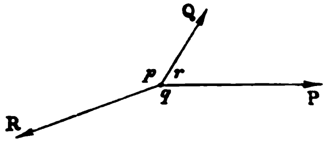

Fig. 5.

Since the sides of the triangle OPO' (Fig. 4) are connected by the equation

$$OO'^2 = OP^2 - 2\,OP.PO' \cos OPO' + PO'^2,$$

we have evidently

$$R^2 = P^2 + 2\,PQ \cos \widehat{PQ} + Q^2,$$

an equation which gives the *magnitude* of the resultant of two forces in terms of the magnitudes of the two forces and the angle between their directions, the forces being represented by two lines, both drawn *from* the point at which they act, as in Art. 17. If $\widehat{PQ} = 0$, the above equation gives $R = P + Q$, or the resultant of two coincident forces is equal to the sum of the forces. If $\widehat{PQ} = \pi$, $R = P - Q$; or, the resultant of two forces which act at a point in exactly opposite senses is equal to the difference of the forces.

21. **Theorem.** *If any one set of forces* (P, Q, R) *acting in three given directions is in equilibrium, all other sets acting in equilibrium in the same directions are merely multiples of the set* (P, Q, R).

For, let the given directions make angles p, q, r, with each other in pairs, and let the sets (P, Q, R) and (P', Q', R') acting in these directions be separate systems in equilibrium. Then we have

$$P : Q : R = \sin p : \sin q : \sin r$$

and

$$P' : Q' : R' = \sin p : \sin q : \sin r.$$

Therefore, $P : Q : R = P' : Q' : R'$, or $\dfrac{P'}{P} = \dfrac{Q'}{Q} = \dfrac{R'}{R}\cdot$ Hence the forces P', Q', R' are separately proportional to P, Q, R, and therefore the former set is not essentially distinct from the latter. This theorem is equivalent to the statement—*when we have determined any one set of forces in equilibrium in three given directions, we have determined all such sets.*

Thus, if we know (see Example 1, Art. 24) that three forces acting along the bisectors of the sides of a triangle drawn from the opposite angles, and proportional to the lengths of these bisectors, are in equilibrium, we know that this is the *only* set in equilibrium in these directions.

Again, we may state the result in the following form, which is known as the PRINCIPLE OF THE TRIANGLE OF FORCES. *If three forces are in equilibrium, and any triangle be drawn with its sides parallel to the lines of action of the forces, the lengths of these sides will represent the magnitudes of the corresponding forces,* on some scale.

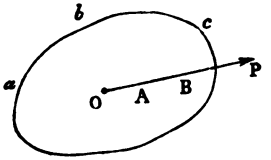

Fig. 6.

22. **Principle of the Transmissibility of Force.** When a force acts on a particle, the force will produce the same effect if it be supposed applied at *any* point along a string connected with the particle, the string lying in the line of action of the force. Thus, if a force equal to the weight of P grammes (Fig. 6) act on a particle, O, in the direction OA,

P may be supposed to act at *A* or *B* at the end of a string attached to *O*. Imagine the particle *O* to be connected with an indefinitely thin rigid membrane, *abc*; then any force *P* acting on *O* may be supposed to be directly applied at *any* point of the membrane in the line of action of *P*.

This axiom is known as the *principle of the transmissibility of force*; it is one of the fundamental principles of Rational Statics, and in most treatises on the subject, it constitutes the basis of the investigation of the conditions of equilibrium. It is essentially necessary to observe that it holds good only for a *rigid* body—that is, a body whose parts, under all circumstances, must maintain constant distances from each other. Thus, if we suppose such a body *about to be* acted on by any set of forces given in magnitudes and directions, we can say, *before the forces are actually applied* at certain points in the body, that the effect will be the same if these forces are applied at *any other points* in their respective lines of action. On the contrary, if the body is deformable, we can make no such assertion. Take, for example, a set of parallel rulers, *ABCD* (Fig. 7), of which the ruler *CD* is fixed, and suppose a force *F* to act on the ruler *AB*, at the point *a*. If, *previous to the action of the force*, it were allowable to transfer its point of application to *b*, on the fixed ruler *CD*, it is clear that the system would remain at rest. But we know that the force *F*, applied at *a*, will cause the ruler *AB* to move until the braces *AD* and *CB* are parallel to the direction of *F*. However, after *the deformable body has taken up a position of equilibrium under the action of the forces*, we can imagine the whole body to be rigidified and each force may then be transferred to any point in its line of action, this point being supposed to be rigidly connected with the system, just as in the case of an indeformable body.

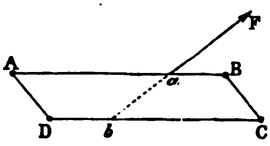

Fig. 7.

Several other very obvious instances of the inapplicability of this principle will doubtless present themselves to the student.

It is essential to observe at the outset that in nature there are no such things as rigid bodies. For a great many practical matters there are bodies which may be treated as if they were rigid or indeformable; but the fact that the particles of solid bodies like iron can be thrown into vibration by the application of even small impulses—as is evidenced by the production of sound from bells and gongs—proves that these bodies are not absolutely rigid.

Bodies which most nearly approximate to the notion of rigidity are called *Natural Solids.*

EXAMPLES.

1. Find the magnitude of the resultant of two forces of 10 kilogrammes' weight and 8 kilogrammes' weight, which act at an angle of 105°.

Ans. $R = 2\sqrt{41-10(\sqrt{6}-\sqrt{2})} = 11{\cdot}06$ kilogrammes' weight.

2. Two forces acting at O are represented by the lines OA and OB; if G is the middle point of AB, the resultant is represented completely by $2.OG$.

3. Two forces, P and Q, of which P is given, act at an angle of 60°; given the magnitude of their resultant, R, find the magnitude of Q.

Ans. $Q = \dfrac{\sqrt{4R^2-3P^2}-P}{2}$.

From this it appears that R cannot be less than $\dfrac{\sqrt{3}}{2}\cdot P$; explain this result by a figure.

4. Two forces, P and Q, inclined at an angle of 120°, have a resultant, R; when they are inclined at an angle of 60°, the resultant becomes n times as great as before; show that

$$P = \frac{R}{2\sqrt{2}}\left(\sqrt{3n^2-1}+\sqrt{3-n^2}\right)$$

$$\text{and } Q = \frac{R}{2\sqrt{2}}\left(\sqrt{3n^2-1}-\sqrt{3-n^2}\right).$$

5. If two forces, acting at a given angle, be each multiplied by the same number, show that their resultant is also multiplied by this number and unchanged in direction.

6. Two forces act at an angle ω; each force becomes n times as great as before, and the angle between the forces is reduced to $\frac{\omega}{2}$; each of these latter forces again becomes n times as great as before, and the angle between them reduced to $\frac{\omega}{4}$. It is observed, that in all these cases the magnitude of the resultant is unaltered. Show that

$$\omega = 4\cos^{-1}\left(\frac{\sqrt{9+4n^2}-1}{4}\right).$$

7. Two chords, OA and OB, of a circle, represent in magnitude and direction two forces acting at the point O; show that if their resultant passes through the centre of the circle, either the chords are equal or they contain a right angle.

8. Find the components of a force, P, along two directions making angles of 30° and 45° with P on opposite sides.

Ans. $\dfrac{2P}{1+\sqrt{3}}$, and $\dfrac{P\sqrt{2}}{1+\sqrt{3}}$.

9. Show that a force represented in magnitude and direction by the diameter of a circle may be resolved into two rectangular components represented by any two rectangular chords of the circle drawn from the extremity of the diameter.

10. Two rectangular forces, P and $P\sqrt{3}$, act on a particle lying on the ground. If P makes an angle of 30° with the horizon, show that the particle will have no horizontal motion.

11. Three forces equal to P, $P+Q$, and $P-Q$, act on a particle in directions mutually including an angle $\dfrac{2\pi}{3}$; find the magnitude and direction of their resultant.

12. O is a point on a circle, and two forces inversely proportional to any two chords, OA and BO, act at O in the senses OA and BO; prove that their resultant acts along the tangent to the circle at O.

(Invert the circle from O; or otherwise.)

13. $ABCD$ is a quadrilateral inscribed in a circle. Along AB and AD, and in these senses, act two forces respectively proportional to the opposite sides, DC, BC; prove that their resultant acts in the diagonal AC and is proportional, on the same scale, to the diagonal BD.

(See Cor. 2, Art. 12.) Hence Ptolemy's Theorem in Geometry.

14. $ABCD$ is a quadrilateral inscribed in a circle. Along AD and BC, and in these senses, act two forces inversely proportional to these chords; prove that if AD and BC intersect in E, and the diagonals AC, BD intersect in O, the resultant of the two forces acts in the line OE.

(See Art. 22, and Cor. 2, Art. 12.)

15. From any point, O, is drawn a right line meeting a circle in A, A', and also another right line meeting the circle in B, B'. If M and N are the middle points of AA' and BB', respectively, and two forces act at O in the senses OM and NO, inversely proportional to these lengths, show that their resultant is perpendicular to the diameter drawn through O.

(Deduce from ex. 12; or otherwise.)

23. Theorem. The following theorem is of wide application in the composition of forces:—

If two forces acting at a point, O, are represented in magnitudes and directions by OB and $n \,.\, OA$, their resultant is represented in magnitude and direction by $(n+1)\ OG$, the point G being taken on AB so that $BG = n \,.\, AG$.

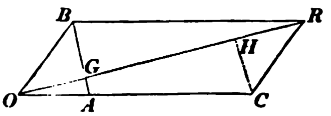

Fig. 8.

For, produce OA to C so that $OC = n \,.\, OA$. Then the two forces acting at O are represented by OC and OB. Complete the parallelogram $OCRB$. Then the diagonal OR is the resultant force.

From C draw CH parallel to AB. Then the triangles CHR and BGO are equal in all respects, $\therefore HR = OG$. Now since $OC = n \,.\, OA$, it follows that $OH = n \,.\, OG$, $\therefore OR = (n+1)\, OG$, which proves the proposition for the magnitude of the resultant.

Again, $\dfrac{CH}{AG} = \dfrac{CO}{OA} = n$, $\therefore CH = n \,.\, AG$, and since $CH = BG$, we have $BG = n \,.\, AG$.

As a particular case, the resultant of two forces represented by OA and OB passes through the middle point of AB, and is represented by twice the line joining O to this point.

If the two forces are equal to $n \,.\, OA$ and $m \,.\, OB$, the resultant passes through the point G determined so that $\dfrac{BG}{AG} = \dfrac{n}{m}$, and is represented on the same scale by $(m+n) \,.\, OG$.

For, diminishing the scale to which the forces are drawn in the ratio of $m : 1$, the two forces will be represented by OB and $\dfrac{n}{m} \cdot OA$. It then follows, by what precedes, that the resultant acts through a point G, such that $BG = \dfrac{n}{m} \cdot AG$, and is equal in magnitude to $\left(\dfrac{n}{m} + 1\right) . \, OG$. If, now, we revert to the original scale, this must be multiplied by m, and we have for the resultant $(n+m) \,.\, OG$.—Q. E. D.

24. Graphic Representation of the Resultant. If several forces, P_1, P_2, . . . act together at a point, their resultant is found thus:—Take the resultant of P_1 and P_2; compounding this resultant with P_3, we get a new force which is the resultant of P_1, P_2, and P_3; compounding this force with P_4, we get the resultant of P_1, P_2, P_3, and P_4; and carrying on this process until all the forces have been used, we obtain in magnitude and direction the resultant of the whole system.

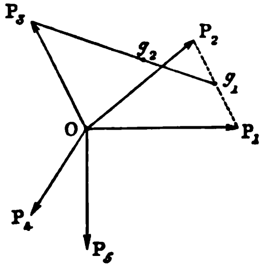

Fig. 9.

Let g_1 be the middle point of the line $P_1 P_2$, which joins the extremities of the first two forces.

Then the resultant of P_1 and P_2 is represented in magnitude and direction by $2 . Og_1$. Compounding the force $2 . Og_1$ with P_3, we get a resultant represented in magnitude and direction by $3 . Og_2$ (Art. 23), where g_2 is a point on $g_1 P_3$ such that $P_3 g_2 = 2 . g_1 g_2$. Again, the resultant of $3 . Og_2$ and P_4 is $4 . Og_3$, where g_3 is the point on $P_4 g_2$ such that $P_4 g_3 = 3 . g_2 g_3$. If there are n forces acting on O, and if G is the last point determined as above, the resultant is represented in magnitude and direction by $n . OG$.

DEF. The point G, thus determined, is called the *Centroid* of the points $P_1, P_2, \ldots P_n$.

COR. 1. If the point O, at which the given forces act, is the centroid of the extremities of the forces $P_1, P_2, \ldots P_n$, the resultant force vanishes, and the particle at O is in equilibrium.

COR. 2. The student will also perceive that if at the points $P_1, P_2, \ldots P_n$ there be placed equal particles, each of mass m, and if each of these particles attracts or repels the particle O with a force proportional to m and to the distances $OP_1, OP_2, \ldots OP_n$, respectively, the resultant attraction or repulsion on O will be $nm . OG$, or $M . OG$, where $M =$ the sum of the masses and G is their centre of mass.

COR. 3. If the attracting or repelling particles form a continuous body, of mass M, and the law of attraction or repulsion is that of the direct distance, the resultant attraction or repulsion will be proportional to $M . OG$, acting in the line OG, where G is the centre of mass of the body.

This result is, therefore, seen to be a simple consequence of the theorem in this Article concerning the resultant of a number of forces acting on a particle—a theorem which was first given by Leibnitz.

EXAMPLES.

1. Find a point inside a triangle such that if a particle placed at it be acted on by forces represented by the lines joining it to the vertices, it will be in equilibrium.

Ans. The intersection of the bisectors of the sides drawn from the opposite angles.

2. $P_1, P_2, \ldots P_n$ are points which divide the circumference of a circle into n equal parts. If a particle, Q, lying on the circumference, be acted upon by forces represented by $QP_1, QP_2, \ldots QP_n$, show that

the magnitude of the resultant is constant wherever Q is taken on the circumference.

It is $n \,.\, QO$, O being the centre of the circle.

3. A particle placed at O is acted on by forces represented in magnitudes and directions by the lines, OA_1, OA_2, ... OA_n, which join O to any fixed points, A_1, A_2, ... A_n; where must O be placed so that the magnitude of the resultant force may be constant?

Ans. If the resultant is represented by a line of length R, O may be placed anywhere on a sphere of radius $\frac{R}{n}$ described round the centroid of the fixed points as centre.

4. Two forces are represented by two semi-conjugate diameters of an ellipse; prove that their resultant is a maximum when the diameters are equal and so taken as to include an acute angle; and that their resultant is a minimum when they are equal and include an obtuse angle.

5. $ABCD$ is a quadrilateral of which A and C are opposite vertices. Two forces acting at A are represented in magnitudes and directions by the sides AB and AD; and two forces acting at C are represented in magnitudes and directions by the sides CB and CD. Prove that the resultant force is represented in magnitude and direction by four times the line joining the middle points of the diagonals of the quadrilateral.

6. O is any point in the plane of a triangle, ABC, and D, E, F are the middle points of the sides. Show that the system of forces OA, OB, OC is equivalent to the system OD, OE, OF. (Wolstenholme, *Book of Mathematical Problems.*)

7. If O be the centre of the circumscribed circle of a triangle, ABC, and L the intersection of perpendiculars from the vertices on the sides, prove that the resultant of forces represented by LA, LB, and LC will be represented in magnitude and direction by $2\,LO$. (Wolstenholme, *ibid.*)

If G is the centroid of the triangle, the resultant is $3\,.\,LG$ (Art. 24); but this, by a well-known theorem in Geometry, is $2\,.\,LO$.

8. Show that the resultant of any number of concurrent forces, P_1, P_2, P_3, ... may be found thus: measure off *any* lengths l_1, l_2, l_3, ... from their point, O, of meeting along their respective lines of action; place at the ends of these lines particles whose masses are proportional to $\frac{P_1}{l_1}, \frac{P_2}{l_2}, \frac{P_3}{l_3}, \ldots$; let G be the centre of gravity of these particles; then OG is the line of action of the resultant of the given forces, and its magnitude is $OG \times \Sigma \frac{P}{l}$. (Mr. Swift P. Johnston.)

25. Graphic Representation of the Resultant. There is another mode of exhibiting the resultant of a number of forces acting on a particle.

When two forces, OA and OB (Fig. 2, p. 7) act at O, their resultant is the diagonal of the parallelogram $OACB$; or, again, it may be considered as the third side of the triangle determined by OA and AC, the latter line being drawn from the extremity of the force OA parallel to the other force, OB.

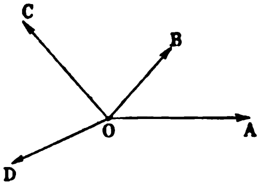

Fig. 10.

Let any number of forces, OA, OB, OC, OD (Fig. 10), act at O. Then drawing oa (Fig. 11) parallel and equal (or proportional) to OA, and from the extremity a drawing ab parallel and equal (or proportional, on the same scale) to OB, the resultant of the forces OA and OB is represented by ob, the third side of the triangle oab. (Of course the resultant acts at O, and is parallel to ob.) Again, drawing bc parallel and equal (or proportional) to OC, the resultant of ob and bc is oc. Compounding this with cd, which represents OD in the above manner, we get the resultant of the whole system represented in magnitude and direction by od, the last side of the polygon $oabcd$.

Hence to represent the resultant of any number of forces acting at a point, O—

Take any point, o, and draw the sides of a polygon successively parallel and proportional to the forces acting at O; then the last side, or that which is required to close up the polygon, represents in magnitude and direction the resultant of the system.

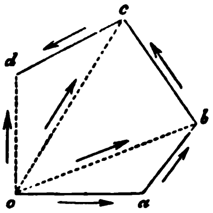

Fig. 11.

Cor. 1. If the last vertex, d, of the polygon of forces closed up into o, the side od would vanish, and the resultant force would vanish; that is, the system of forces would be in equilibrium. Hence—

If the sides of a closed polygon marked with arrows, which all go round the polygon IN THE SAME SENSE, *represent in magnitudes and directions the forces which act together on a particle, these forces form a system in equilibrium.*

Cor. 2. When only three forces act, the preceding Cor. shows that they will be in equilibrium if they are parallel and proportional to the sides of a triangle which are marked with arrows all going round the triangle in the same sense.

This proposition has been already enunciated as the *Triangle of Forces*.

26.] LAPLACE'S PROOF OF THE PARALLELOGRAM OF FORCES. Among purely statical proofs of this fundamental proposition, i.e. proofs which do not depend on the consideration of velocity, Laplace's appears to be the most elegant, and as, moreover, it does not involve the principle of transmissibility, it is thought desirable to include it in the present treatise.

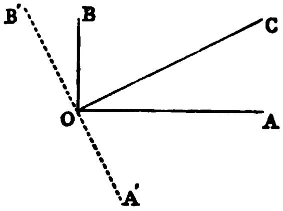

Fig. 12.

Let two rectangular forces, P and Q, represented by the lines OA and OB (Fig. 12), act at O, and let R be the unknown magnitude, and OC the unknown direction, of their resultant. It is evident that if P and Q give a resultant equal to R acting in OC, nP and nQ will give a resultant equal to nR acting also in OC, because taking multiples of the forces is the same thing as merely altering the scale of magnitude to which they are referred. Conversely, whatever n may be, nR may be replaced by nP, making an angle $\theta\,(= COA)$, and nQ, making an angle $\frac{\pi}{2}-\theta\,(=COB)$ with the direction of R. Let n be taken $=\frac{P}{R}$ and draw $A'OB'$ perpendicular to OC. Then since

$$R \text{ may be replaced by } P \text{ in } OA \text{ and } Q \text{ in } OB,$$

$$P \quad ,, \quad ,, \quad ,, \quad \frac{P^2}{R} \text{ in } OC \quad ,, \quad \frac{PQ}{R} \text{ in } OA'.$$

Similarly Q may be replaced by $\frac{PQ}{R}$ in OB' and $\frac{Q^2}{R}$ in OC.

Hence the forces P and Q are equivalent to a force $=\frac{P^2}{R}+\frac{Q^2}{R}$ in OC, a force $\frac{PQ}{R}$ in OA', and a force $\frac{PQ}{R}$ in OB'. But these last are equal and opposite, and therefore they destroy each other. Hence P and Q are equivalent to a single force $=\frac{P^2+Q^2}{R}$ acting in the direction of their resultant; therefore

$$R=\frac{P^2+Q^2}{R},$$

or

$$R=\sqrt{P^2+Q^2}. \qquad (1)$$

Thus we have found the *magnitude* of the resultant of any two rectangular forces. We now proceed to find its direction.

If P and Q are equal, their resultant bisects the angle between them, and (1) therefore shows that it is represented in magnitude and direction by the diagonal of their parallelogram.

Let three forces, at right angles to each other, OA, OB, and OC (Fig. 13) each equal to P, act on a particle O; complete the cube as in the figure. By what precedes, the resultant of OB and OC is OF; combining this with OA, we see that the direction of the resultant lies in the plane FOA. Similarly, it can be proved to lie in the plane COD; hence its direction is OO', the intersection of these planes, or the diagonal of the cube. Now from (1) $OF = P\sqrt{2}$, and the resultant of the three forces is the same as the resultant of $P\sqrt{2}$ along OF and P along OA. By (1) the magnitude of the resultant is $P\sqrt{3}$, and since $OO' = P\sqrt{3}$, we have proved that the diagonal, OO', of the parallelogram FOA represents in magnitude and direction the resultant of two forces P and $P\sqrt{2}$.

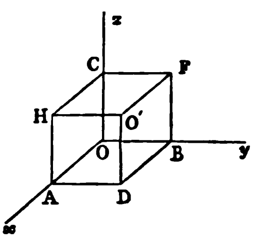

Fig. 13.

Suppose now that $OA = P$, $OB = P\sqrt{2}$, and $OC = P$, and complete the parallelopiped. We have just proved that the resultant of $OB (= P\sqrt{2})$ and $OC (= P)$ is the diagonal OF $(= P\sqrt{3})$; and since the resultant of the three forces must lie in the planes COD and FOA, it must act in the diagonal OO'. But this resultant is the resultant of $P\sqrt{3}$ along OF and P along OA, and by (1) its magnitude is $P\sqrt{4}$, which is the magnitude of OO', the diagonal of the parallelogram FOA.

By keeping OA and OC each equal to P, and giving OB the values $P, P\sqrt{2}, P\sqrt{3}, \dots P\sqrt{m}$, successively, we prove in this way that the parallelogram law holds for P and $P\sqrt{m}$, where m is any integer. Again, keeping $OB = P\sqrt{m}$, $OC = P$, and making $OA = P, P\sqrt{2}, P\sqrt{3}, \dots P\sqrt{n-1}$, in succession,—where n is an integer,—we prove that the law holds for $P\sqrt{m}$ and $P\sqrt{n}$, i.e., for any two rectangular forces the squares of whose magnitudes are proportional to two integers; hence it holds for any two rectangular forces, because even if their ratio involves an endless decimal, this (and its square) can be expressed, to any desired degree of accuracy, as the ratio of an integer number to some power of 10. The parallelogram law, then, holds for two rectangular forces which bear to each other any given ratio.

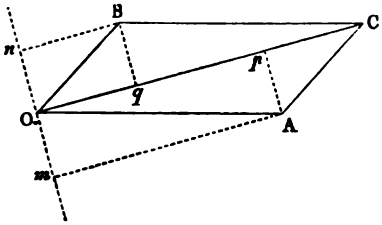

Fig. 14.

From this the proposition follows easily for oblique forces.

Let *OA* and *OB* (Fig. 14) represent two oblique forces, *P* and *Q*; complete the parallelogram, draw the line *mn* through *O* perpendicular to the diagonal *OC*, and let fall the perpendiculars *Ap*, *Am*, *Bq*, and *Bn*, on *OC* and *mn*. By what we have proved, the force *OB* (= *Q*) can be replaced by *Oq* and *On*, and *OA* (= *P*) can be replaced by *Op* and *Om*. But *Om* is evidently equal and opposite to *On*, therefore *OC* is the line of action of the resultant, and its magnitude = $Op + Oq$, which = *OC*.

CHAPTER II.

GENERAL CONDITIONS OF THE EQUILIBRIUM OF A PARTICLE UNDER THE ACTION OF FORCES IN ONE PLANE.

27.] **Absolute Condition of Equilibrium.** One condition is necessary and sufficient for the equilibrium of a particle—and that condition is, that *the magnitude of the resultant force acting upon it shall be zero.* In the case of a body of finite extension (as distinguished from a mere particle) the student will afterwards see that this single condition is not sufficient. The vanishing of the Resultant may be called the *absolute* condition of the equilibrium of a particle.

28.] **Several Forces.** When several forces act upon a particle, the condition of its equilibrium may be expressed as in Cor. 1, p. 21 ; or as in Cor. 1, p. 23. But in practice, these representations would frequently be found clumsy, and we obtain simpler results by using the principle of the Resolution of Forces than those given by the principle of Composition. It is to be observed that forces acting on a particle are to be considered as *forces whose lines of action all pass through one common point.*

29.] **Resolution of Forces in given Directions.** It has been proved that a force can be resolved into two others along any two directions in the same plane, if this plane contains the line of action of the force. Simplicity is gained by taking these two directions at right angles to each other. Thus, let Ox and Oy be any two lines at right angles to each other, and P any force acting at O in the plane Oxy. Then, completing the parallelogram $OXPY$, we find the components, OX and OY, of the force P along the axes Ox and Oy. Let OX and OY be denoted simply by X and Y. It is, then, evident that

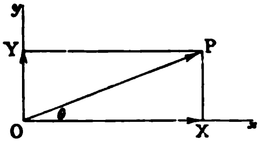

Fig. 15.

$$X = P\cos\theta,$$
$$Y = P\sin\theta,$$

where θ is the angle which the direction of P makes with Ox.

In strictness, when we speak of the component of a given force along a certain line, it is necessary to mention the other line along which the other component acts. For example, the force P may have an infinite number of components along the same right line Ox. If the line associated with Ox be Om, and if the parallelogram $OMPM'$ be completed, the component of P along Ox will be OM, the other component being OM'. If, again, the resolution of P be effected along Ox and On, and the parallelogram $ONPN'$ be drawn, the component of P along Ox will be ON; and it is evident that if ω be the angle between the axes along which P is resolved, the component along Ox will be $P\cdot\dfrac{\sin(\omega-\theta)}{\sin\omega}$.

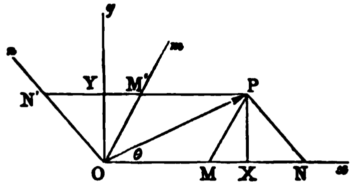

Fig. 16.

In what follows, unless the contrary is expressed, by the component of a force along any line we shall understand the *rectangular component*; that is, the resolution is supposed to be made along this line and the line perpendicular to it. It must be remembered, then, that—

The component of a force, P, along a right line is $P\times\cos$ (angle between right line and direction of P).

30.] Equations of Equilibrium, or Analytical Conditions. If several forces, P_1, P_2, P_3, ... act at O, each of them may be replaced by its two components, one along Ox, and the other along Oy, which is perpendicular to Ox (Fig. 17). Thus, the components of P_1 are $P_1\cos\theta_1$, and $P_1\sin\theta_1$; those of P_2 are $P_2\cos\theta_2$, and $P_2\sin\theta_2$, and these latter are measured in exactly the same senses as the components of P_1; that is to say, $P_2\cos\theta_2$ is the component of P_2 along Ox in the sense Ox. The component of P_2 in the figure is actually in the sense *opposite* to Ox, that is, in the sense $O, -x$; still,

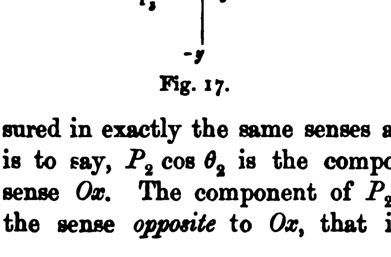

Fig. 17.

the component in the sense Ox is $P_2 \cos \theta_2$, for $\cos \theta_2$ is negative. If the senses Ox and Oy are regarded as the positive senses, any components which act in the opposite senses, $O, -x$ and $O, -y$, would subtract from the positive components, and must be considered negative. It will be seen that the negative sign of every component will be perfectly represented and accounted for by the general expressions, $P \cos \theta$ and $P \sin \theta$, for the two components. Thus, the figure shows that both components of P_3 are negative, and accordingly both of the expressions $P_3 \cos \theta_3$ and $P_3 \sin \theta_3$ are negative, since θ_3 is $> 180°$.

In order that the expressions $P \cos \theta$ and $P \sin \theta$ may always represent components in the positive senses Ox and Oy, the angle θ must be measured from Ox towards the line of action of the force in a fixed sense—that opposite to watch-hand rotation being generally chosen.

With this understanding, then, we may say that the components of P_1, P_2, P_3 in the direction Ox are $P_1 \cos \theta_1$, $P_2 \cos \theta_2$, and $P_3 \cos \theta_3$, and those in the direction Oy are $P_1 \sin \theta_1$, $P_2 \sin \theta_2$, and $P_3 \sin \theta_3$.

Replacing each of the forces, P_1, P_2, P_3, ... by its components, we have

$$P_1 \cos \theta_1 + P_2 \cos \theta_2 + P_3 \cos \theta_3 + \ldots, \text{ or } \Sigma P \cos \theta, \text{ along } Ox,$$

and

$$P_1 \sin \theta_1 + P_2 \sin \theta_2 + P_3 \sin \theta_3 + \ldots, \text{ or } \Sigma P \sin \theta, \text{ along } Oy.$$

If the component, $P \cos \theta$, of a force, P, along Ox, be denoted by X, and that along Oy by Y, the whole system of forces is equivalent to the two single forces,

$$X_1 + X_2 + X_3 + \ldots, \text{ or } \Sigma X, \text{ along } Ox,$$

and

$$Y_1 + Y_2 + Y_3 + \ldots, \text{ or } \Sigma Y, \text{ along } Oy.$$

Now, since (Art. 26, p. 24) the resultant of two forces, P and Q, at right angles is $\sqrt{P^2 + Q^2}$, the resultant, R, of the system of forces P_1, P_2, ..., is given by the equation

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}. \qquad (1)$$

For the equilibrium of O it is necessary and sufficient that $R = 0$. Hence

$$(\Sigma X)^2 + (\Sigma Y)^2 = 0. \qquad (2)$$

Now, this equation cannot be satisfied, so long as ΣX and ΣY are real quantities, unless

$$\Sigma X = 0 \text{ and } \Sigma Y = 0. \qquad (3)$$

These, then, are the two necessary and sufficient conditions for the equilibrium of the particle, and they are equivalent to the *single* condition $R = 0$. (See Art. 27.)

The equations (3) are equivalent to the following statement:—

For the equilibrium of a particle acted on by any number of forces in one plane, it is necessary and sufficient that *the algebraic sum of the rectangular components of the forces, along each of two right lines at right angles to each other in the plane of the forces, should vanish.* Since the directions Ox and Oy, along which the forces are resolved, may be any whatever in their plane, we may evidently vary the above statement thus—*the algebraic sum of the rectangular components of the forces along every right line in their plane is zero.*

It is merely for uniformity of notation that we have measured $\theta_1, \theta_2, \theta_3, \ldots$ (Fig. 17), all in the same sense—that opposite to watch-hand rotation. In resolving forces along a line, Ox, it is simpler in practice to use the acute angles made by the forces with the line, and to indicate negative components by the sign *minus*.

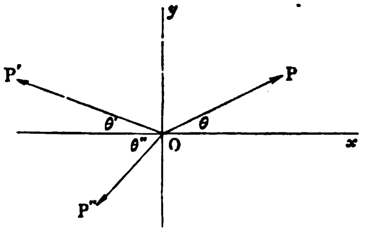

Fig. 18.

Thus, if (Fig. 18) the forces P, P', P'' make acute angles $\theta, \theta', \theta''$, with Ox, the sum of the components of the forces along Ox is

$$P \cos\theta - P' \cos\theta' - P'' \cos\theta'',$$

and that along Oy is

$$P \sin\theta + P' \sin\theta' - P'' \sin\theta''.$$

The rectangular component of a force along a line is sometimes called the *effective* component along this line.

Cor. A force has no effective component in a direction at right angles to itself.

31.] **Direction of the Resultant.** The direction of the resultant of any number of forces acting in one plane on a particle, O, is known when its components, ΣX and ΣY, along any two directions, Ox and Oy, are known. For, if Ox and Oy are rectangular, and a be the angle which the resultant, R, makes with Ox, we have, evidently (Fig. 19),

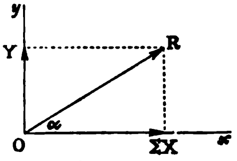

Fig. 19.

$$\tan a = \frac{\Sigma Y}{\Sigma X}; \qquad (4)$$

and if Ox and Oy include an angle ω,

$$\frac{\sin a}{\sin(\omega - a)} = \frac{\Sigma Y}{\Sigma X}.$$

32.] **Tension of a String.** When a string is employed to connect two or more particles which are acted on by given forces, the fibres of the string become subject to a certain pull, stress, or *tension*, which, if increased beyond a certain limit, will cause the string to break. This tension is a force *which at any point of the string may be conceived as acting in either of two opposite senses, or in both of these senses at once*, according to the nature of the question under discussion. Let us consider, as a simple example, the case of a string, AB (Fig. 20), whose weight we may neglect, fixed at the extremity A, and attached at B to a mass of weight W. If, now, we imagine the string to be cut at any point p, and the lower portion, pB, to be removed, it is clear that the remaining portion, pA, will not be in the same state of stress as before unless we apply at the section p a force equal to W, and acting downwards. Again, let the string be cut a little above p, at q, and suppose the portion qA removed. Then the small portion, pq, will not remain in its place unless an upward force equal to W is applied at the section q. The small portion of the string included between p and q is then kept at rest by two equal and opposite forces, each equal to W. Thus, then, if we consider any portion, pq, as isolated from the rest of the string, we must represent it as subject to two equal tensions directly opposed to each other. If we considered the action of the upper portion, pA, on the lower, pB, we should represent pB as acted on by an upward force applied at p; and if we consider

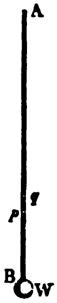

Fig. 20.

the action of the lower on the upper, we must represent *pA* as acted on by a downward force applied at the section of separation of *pA* and *pB*. Thus, the action at *B* of the string on the body *W* is an upward force, or tension, equal to *W*; while the action of *W* on the string consists of an equal force in the opposite direction.

33.] **String passing over smooth pegs or surfaces.** When a string whose weight we neglect passes over a smooth peg, or over any number of smooth surfaces, we shall assume for the present that the stress of its fibres, or its tension, is the same at all of its points. Should it, however, be *knotted* at any of its points to other strings, we must regard its continuity as broken, and the tension will not be the same in the two portions which start from a knot. Thus, if the string at rest pass over two smooth surfaces, *A* and *B* (Fig. 21), and if it is pulled at one extremity by a force *P*, it must be pulled at the other extremity with an equal force; but if, after leaving the surface *A*, it is knotted at *C* to another string which is pulled with a force equal to *R*, the tensions in the portions between *C* and *A* and between *C* and *B* are no longer the same, and their relative magnitudes must be determined by equation (*a*) of Chap. I., Art. 20.

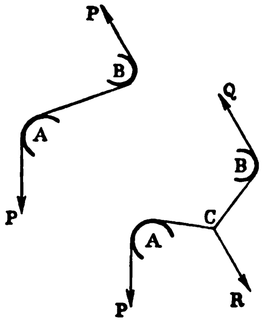

Fig. 21.

34.] **Equilibrium of a System of Particles.** When several particles are connected together and form a system, each particle being acted upon by special forces in addition to the forces produced upon it by its connection (by strings or rods) with the other particles, we can consider the equilibrium of any *one* particle apart from all the others, *provided that we take account of all the forces which are produced on it by its connection with the others, in addition to the special forces acting upon it.*

This may be called the *principle of separate equilibrium*, which is continually employed in all branches of Dynamics.

Thus, in No. 8 of the following examples, we may write down equations for the equilibrium of the particle *N* as if it were entirely disconnected with the other points, *A*, *P*, *M*, *B*, if we represent it as acted on by the force, W, and by the tensions, T_2 and T_3, of the strings by which it is connected with the system.

35. **Numerical Calculation.** When in any instance the numerical values of forces are assigned, the student will derive much benefit from the practice of constructing good figures, which, with as much fidelity as is practically attainable, truly represent the relative magnitudes and directions of the forces. For this purpose, the first thing to be done is to fix on some convenient *scale* of representation. The scale to be adopted will, as already pointed out in Art. 17, depend on the magnitudes assigned in each particular case. Thus, if the student works on "section paper," he may take one, two, three, or more of the sides of its little squares to represent the unit force. If the magnitudes of the forces given range from, say, 5 to 20 kilogrammes' weight, it will be convenient enough to take a side of one square to represent 1 kilogramme weight; but if the forces range only from 5 to 10, two, or even three, sides may be taken to represent 1 kilogramme weight; for the greater the length which represents the unit force, the less the error which is likely to occur in the drawing. In fact, without trigonometrical calculation, the magnitudes of certain required forces may be found by carefully drawing the forces that are given, and *measuring* with instruments the lines which represent the required forces, the results being reliable to the first place of decimals. Several of the following examples are suitable to such a method, and the student is recommended to *verify* his calculated results by subsequent measurement.

36.] **Useful Trigonometrical Theorem.** The following theorem will be found extremely useful in the solution of a large number of statical problems.

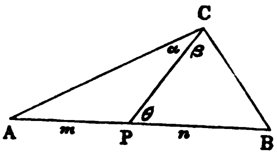

Fig. 22.

If a right line CP (Fig. 22) drawn from the vertex of a triangle divide the base into two segments m and n, or segments which are to each other in the ratio $m : n$,

$$(m+n)\cot\theta = m\cot\alpha - n\cot\beta, \tag{1}$$

where α and β are the angles which CP makes with CA and CB, and θ is the angle which CP makes with the base AB.

For, if $AP = m$, $BP = n$,

$$CP = m\frac{\sin A}{\sin \alpha} = m\frac{\sin (\theta - \alpha)}{\sin \alpha} = m(\sin\theta\cot\alpha - \cos\theta).$$

Also $$CP = n\frac{\sin B}{\sin \beta} = n\frac{\sin (\theta + \beta)}{\sin \beta} = n(\sin\theta\cot\beta + \cos\theta).$$

Equating these two values of CP, we obtain (1) at once.

The following equation also holds:

$$(m+n)\cot\theta = n\cot A - m\cot B. \tag{2}$$

For, $$CP = m\frac{\sin A}{\sin \alpha} = m\frac{\sin A}{\sin (\theta - A)} = \frac{m}{\sin\theta\cot A - \cos\theta}.$$

Similarly $CP = \dfrac{n}{\sin\theta\cot B + \cos\theta}$. Equating these values of CP, (2) follows at once.

COR. If A and B are two fixed points and C any variable point, and if the angles CAB and CBA are denoted by θ and ϕ, respectively, any equation of the form

$$n\cot\theta - m\cot\phi = k, \tag{a}$$

where n, m, and k are constants, will be satisfied if C has any position on a certain fixed right line—viz., a line dividing AB in P so that $AP : BP = m : n$, and making with AB an angle, CPB or γ, such that

$$\cot\gamma = \frac{k}{m+n},$$

or in other words, any equation of the form (a) denotes a rectilinear locus—as is evident also from the elements of analytic geometry.

Frequent reference will be made to this result in the sequel.

EXAMPLES.

1. At the point, O, of intersection of diagonals of a square (Fig. 23), let two forces of 8 and 12 grammes' weight, act along the diagonals, and two forces of 10 and 2 grammes' weight, act perpendicularly to two sides; required the magnitude and direction of their resultant.

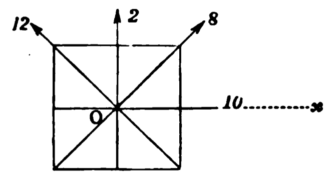

Fig. 23.

Resolving the forces along Ox, the line of action of one of them, the component of the force 10 is 10, that of the force 8 is 8 cos 45°, that of 2 is zero, and that of 12 is -12 cos 45°. Hence

$$\Sigma X = 10 + \frac{8}{\sqrt{2}} - \frac{12}{\sqrt{2}} = 10 - 2\sqrt{2}.$$

Similarly, $\Sigma Y = \frac{8}{\sqrt{2}} + 2 + \frac{12}{\sqrt{2}} = 2 + 10\sqrt{2}.$

Therefore $R = \sqrt{(10-2\sqrt{2})^2 + (2+10\sqrt{2})^2} = \sqrt{312}.$

Again, if a be the angle made by R with Ox,

$$\tan a = \frac{2+10\sqrt{2}}{10-2\sqrt{2}} = \frac{1+5\sqrt{2}}{5-\sqrt{2}} = 2\tfrac{1}{4} \text{ (nearly)}.$$

2. Three forces, P, Q, R, act on a particle: find the magnitude of their resultant.

Let the angles opposite P, Q, and R be denoted by p, q, r (Fig. 5, p. 15). Then resolving all the forces along the direction of P, we get for their combined component in this direction $P + Q\cos r + R\cos q$. Resolving them perpendicularly to P, their total component $= Q\sin r - R\sin q$. Hence the square of the resultant $= (P + Q\cos r + R\cos q)^2 + (Q\sin r - R\sin q)^2$. Remembering that $p+q+r = 2\pi$, this is easily seen to be

$$P^2 + Q^2 + R^2 + 2PQ\,.\cos r + 2\,QR\,.\cos p + 2RP\,.\cos q.$$

3. Verify in the last question that if the three forces are in equilibrium, the expression given for the resultant vanishes.

When the forces are in equilibrium,

$$P : Q : R = \sin p : \sin q : \sin r.$$

Hence the expression for the square of the resultant is proportional to

$$\sin^2 p + \sin^2 q + \sin^2 r + 2\sin p \sin q \cos r + 2\sin q \sin r \cos p + 2\sin r \sin p \cos q.$$

The last two terms $=$

$$2\sin r \sin(p+q) = -2\sin^2 r, \quad \because p+q = 2\pi - r.$$

Therefore the expression is

$$\sin^2 p + \sin^2 q - \sin^2(p+q) + 2\cos(p+q)\sin p \sin q = \sin^2 p + \sin^2 q - 1 + \cos(p+q)\cos(p-q), \quad \because 2\sin p \sin q = \cos(p-q) - \cos(p+q).$$

Now,

$$\cos(p+q)\cos(p-q) = 1 - \sin^2 p - \sin^2 q,$$

$\therefore$ the square of the resultant $= 0$.

4. A heavy particle at O (Fig. 24), whose weight is W, is held in equilibrium by three forces (in addition to its weight)—$\frac{W}{n}$ acting horizontally, F acting in a direction making an angle i with the horizon, and R at right angles to F; find the magnitudes of F and R in terms of the given force W.

Resolve all the forces along the directions of F and R successively. These directions are chosen rather than any others, because, since R is at right angles to F, it will give no component along F, and, for the same reason, F will give no component along R.

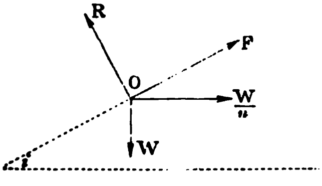

Fig. 24.

The component along OF is $F + \frac{W}{n}\cos i - W\sin i$.

For equilibrium it is necessary (Art. 30, equations (3)) that this component shall be zero. Hence

$$F + \frac{W}{n}\cos i - W\sin i = 0,$$

$$\therefore F = W\left(\sin i - \frac{1}{n}\cos i\right).$$

Again, the sum of the components along OR is

$$R - W\cos i - \frac{W}{n}\sin i;$$

and this must also be zero. Hence

$$R = W\left(\cos i + \frac{1}{n}\sin i\right).$$

The same values would, of course, be found if we had selected any two other directions for the resolution. Thus, if we resolve all the forces vertically, i.e., in the direction OW, we get

$$W - F\sin i - R\cos i = 0;$$

and resolving horizontally, or in the direction of $\frac{W}{n}$, we get

$$\frac{W}{n} + F\cos i - R\sin i = 0.$$

Solving these last two equations for R and F, we get the same values as before.

The advantage of a judicious selection of directions for the resolution of the forces is now apparent. By resolving at right angles to one of the unknown forces, we obtained an equation free from that force; whereas, when the directions were selected at random, both of the unknown forces entered into each of our equations, and to find these forces it was then necessary to solve the equations.

Having selected one direction for resolution, it is not necessary that the second should be selected at right angles to it; for the student has seen (p. 30) that when a particle is in equilibrium, the sum of the components of the forces along *any direction whatever* must be zero. Hence we might, in the present case, have resolved vertically and along the direction OF, and the equations thus obtained would have given the same results as before.

5. One end of a string is attached to a fixed point, A (Fig. 25); the string, after passing over a smooth peg, B, sustains a given mass, P, at its other extremity, and to a given point, C, in the string is knotted a particle of given weight, W. Find the position of equilibrium of the system.

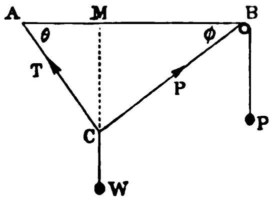

Fig. 25.

Before setting about the solution of statical problems of this kind, the student will clear the ground before him, and greatly simplify his labour by asking himself the following questions :—

(*a*) What lines are there in the figure whose lengths are already given ?

(*b*) What forces are there whose magnitudes are already given, and what are the forces whose magnitudes are as yet unknown ?

(*c*) What variable or variables in the figure would, if it or they were known, determine the required position of equilibrium ?

Now, in the present case (*a*), the linear magnitudes which are given are the lengths AB and AC. The entire length of the string is of no consequence, since it is clear that, once equilibrium is established, P might be suspended from a point at any distance whatever from B. Again (*b*), the forces acting at the point C are the weight, W, a tension in the string CA, and another tension in the string CB. Of these, W is given, and so is the tension in CB, which must, since the peg is smooth, be equal to P (see Art. 33); but there is, as yet, nothing determined about the magnitude of T, the tension in CA. And (*c*) the angle, θ, of inclination of the string CA to the horizon would, if known, at once determine the position of equilibrium. For, if θ is known, we draw AC of the given length: then, joining C to B, the position of the system is completely known. The angle, ϕ, of inclination of BC to the horizon, would do equally well; and it is evident that, since either angle suffices, each must be capable of being expressed in terms of the other, and the given magnitudes in the question.

Let $AB = a$, $AC = b$. Then, for the equilibrium of the point C we have, by equation (*a*), p. 15,

$$\frac{P}{W} = \frac{\cos\theta}{\sin(\theta+\phi)}. \tag{1}$$

To this equation must be joined the relation between θ and ϕ given by the geometry of the figure. We have, evidently,

$$AC \,.\, \sin ACB = AB \,.\, \sin\phi,$$

or

$$b\sin(\theta+\phi) = a\sin\phi. \tag{2}$$

Equation (1) gives

$$\frac{a\sin\phi}{b\cos\theta} = \frac{W}{P}, \tag{3}$$

or

$$\sin\phi = \frac{bW}{aP}\cos\theta,$$

$$\therefore \cos\phi = \frac{\sqrt{a^2P^2 - b^2W^2\cos^2\theta}}{aP}.$$

Expanding $\sin(\theta+\phi)$ in (2), and substituting these values of $\sin\phi$ and $\cos\phi$, and reducing, we have the equation

$$\cos^3\theta - \frac{P^2a^2 + W^2(a^2+b^2)}{2abW^2}\,.\,\cos^2\theta + \frac{P^2a}{2W^2b} = 0. \tag{4}$$

We may obtain this result very simply by employing a Triangle of Forces. Thus, from M draw a line MQ parallel to BC, meeting AC in Q. Then MCQ is a triangle of forces for the point C, its sides being parallel to the three equilibrating forces at C. Hence $\frac{P}{W} = \frac{QM}{MC}$; but $QM = \frac{AM}{AB} . BC = \frac{b}{a} \cos\theta (a^2 - 2ab\cos\theta + b^2)^{\frac{1}{2}}$; and $MC = b\sin\theta$;

$$\therefore \frac{P}{W} = \frac{\cos\theta \, (a^2 - 2ab\cos\theta + b^2)^{\frac{1}{2}}}{a\sin\theta},$$

which is the same as the previous equation for θ.

When the numerical values of the constants are assigned, (4) becomes an equation with numerical coefficients, the required root of which can be obtained (by Horner's method, for instance) to any desired degree of accuracy.

A tolerably accurate solution can also be readily obtained by a graphic method thus: describe round A the circle of radius b, and on AB take a point M such that $AM : MB = Wb : Pa$; describe circles on AM and MB as diameters. Then C must be found on the circle of radius b so that if CA meets the circle on AM in L, and if CB meets the circle on MB in N, the lines

AL and MN are equal.

This will give, by scale measurement, approximate values of θ and ϕ. If these are α, β, respectively, we can put $\theta = \alpha + x$, $\phi = \beta + y$, where x, y are two very small circular measures. Substituting these values of θ, ϕ in (2) and (3), and retaining only the first powers of x and y, we have two simple equations for these unknown corrections.

The student will do well to observe that the coefficients of equation (4) are ratios of magnitudes of the same kind. Force and linear magnitude are quantities of essentially different kinds. It is true, indeed, that the magnitude of a force may be conventionally represented by the length of a line, but it is only in comparison with other forces that any one force can be so represented, and the scale of representation is arbitrary. Hence $\cos\theta$, which is a mere number, if it is expressed in terms of force, must be expressed as the *ratio of one force to another*; and if it is expressed in terms of linear magnitude, it must be as the *ratio of one line to another*. If, for example, the coefficient of $\cos^3\theta$ in (4) being unity, the last term had been $\frac{Pa^3}{W^2b}$, we should have known at once that the result was wrong. For the numerator and denominator of this expression are not of the *same degree* in force; neither are they of the same degree in linear magnitude. Such a term as $\frac{Pa^3}{W^2b}$ denotes the product of an area, $\frac{a^3}{b}$, by the reciprocal of a force, $\frac{P}{W^2}$. If for some unknown quantity of any kind (force, length, trigonometrical function, etc.) we obtain a value which violates the *principle of homogeneity*, then, by a mere alteration of the units in terms of which the given constants are expressed, we obtain new and inconsistent values of the quantity sought—as the student will at once see by taking some special example.

Similar remarks as to the homogeneity of our results will be of frequent occurrence in the sequel. By attention to considerations of this kind the student will often be able to detect an error in his work.

6. If, in the last example, the mass W, instead of being knotted to the string at C, is suspended from a smooth ring which is at liberty to slide along the string ACB, find the position of equilibrium.

In this case, the string $PBCA$, which passes over a smooth surface at B, and through the smooth ring, will have its tension constant at each of its points (Art. 33), and therefore equal to P. Hence, putting $T = P$, and resolving forces vertically for the equilibrium of C, we have

$$W - 2P\sin\theta = 0,$$

or

$$\sin\theta = \frac{W}{2P}.$$

7. A string, whose weight is neglected, passes over three smooth pegs, A, B, C, which are in the same horizontal line. From the extremities of the string are suspended two masses, P and P'; and to two given points in it are knotted two masses, W and W', the first suspended between A and B, and the second between B and C. Find the position of equilibrium.

In this problem the given quantities are the weights of the masses P, W, P', and W', the distances AB and BC, and the length of the portion mBm' of the string (Fig. 26).

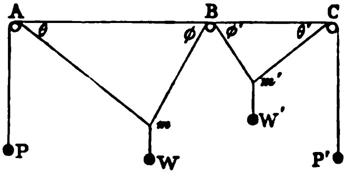

Fig. 26.

Evidently the quantities which we wish to determine are the inclinations, θ, ϕ, ..., of the portions of the string to the horizon.

Let $AB = a$, $BC = a'$, and the length of $mBm' = k$. Consider the equilibrium of the point m. Since the string PAm passes over a smooth peg at A, the tension in it $= P$ throughout. If $T =$ tension in mBm', we have for the equilibrium of m,

$$\frac{P}{W} = \frac{\cos\phi}{\sin(\theta+\phi)}. \qquad (1)$$

$$\frac{T}{W} = \frac{\cos\theta}{\sin(\theta+\phi)}\cdots.$$

Again, for the equilibrium of m',

$$\frac{P'}{W'} = \frac{\cos\phi'}{\sin(\theta'+\phi')}. \qquad (2)$$

$$\frac{T}{W'} = \frac{\cos\theta'}{\sin(\theta'+\phi')}.$$

Equating the two values of T, we have

$$\frac{W\cos\theta}{\sin(\theta+\phi)} = \frac{W'\cos\theta'}{\sin(\theta'+\phi')}. \qquad (3)$$

These are all the equations that can be obtained from statical considerations. One more equation is required to determine the four unknown quantities, θ, ϕ, θ', and ϕ'. This is obtained by expressing that the length of $mBm' = k$. Evidently

$$Bm = \frac{a\sin\theta}{\sin(\theta+\phi)}, \text{ and } Bm' = \frac{a'\sin\theta'}{\sin(\theta'+\phi')};$$

$$\therefore \frac{a\sin\theta}{\sin(\theta+\phi)} + \frac{a'\sin\theta'}{\sin(\theta'+\phi')} = k. \qquad (4)$$

These four equations determine θ, ϕ, θ', ϕ', and therefore the position of equilibrium.

8. A string, $BMNP \ldots A$, whose weight is neglected, is suspended from two fixed points, A and B; and from given points, M, N, P, ..., in the string, are suspended a series of equal particles, the weight of each being W. Find the inclinations, θ_1, θ_2, θ_3, ..., of the successive portions of the string to the horizon.

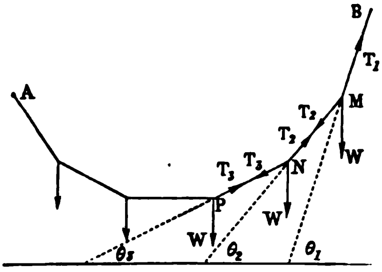

Fig. 27.

Consider the equilibrium of the particle at M. This particle is kept in equilibrium by three forces, viz., W acting vertically, T_1, the tension of the string MB, and T_2, the tension of MN.

Resolving these forces vertically,

$$W + T_2\sin\theta_2 - T_1\sin\theta_1 = 0; \qquad (1)$$

and, resolving horizontally,

$$T_1\cos\theta_1 - T_2\cos\theta_2 = 0.$$

For the equilibrium of N, resolving horizontally,

$$T_2\cos\theta_2 - T_3\cos\theta_3 = 0.$$

Hence

$$T_1\cos\theta_1 = T_2\cos\theta_2 = T_3\cos\theta_3 = \ldots\ldots;$$

or in other words, *the horizontal components of the tensions in the different portions of the string are constant.* Let this constant be denoted by T; then

$$T_1 = \frac{T}{\cos\theta_1},\ T_2 = \frac{T}{\cos\theta_2},\ \&c.$$

Substituting these values in (1), we have

$$\tan\theta_1 = \tan\theta_2 + \frac{W}{T}.$$

Similarly,

$$\tan\theta_2 = \tan\theta_3 + \frac{W}{T},$$

$$\tan\theta_3 = \tan\theta_4 + \frac{W}{T},$$

. . . .

Hence *the tangents of the successive inclinations form a series in Arithmetical Progression.* In the figure

$$\theta_4 = 0,\ \therefore \tan\theta_3 = \frac{W}{T},\ \tan\theta_2 = \frac{2W}{T},\ \tan\theta_1 = \frac{3W}{T}.$$

If the suspended weights are not equal, it is still true that the horizontal components of the tensions are all equal.

The figure formed by the string $BMNP \ldots A$ is called a *Funicular Polygon*, i. e., a rope polygon; but the term is also used in a much more extended sense, as will be seen later.

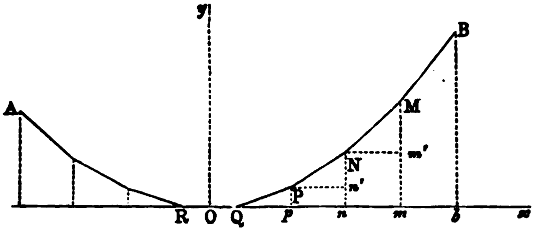

Fig. 28.

9. To construct the Funicular Polygon, when the horizontal projections, RQ, Qp, pn, nm, mb, . . . , of the successive portions of the chain are all of the same length, a, and one portion is horizontal.

Let $Pp = c$; then, since (last example) the tangent of the inclination of PN = 2 . tangent of inclination of PQ, it follows that, Pn' being horizontal, $Nn' = 2\,Pp = 2c$. Also tan of inclination of MN = 3 tan of inclination of PQ; $\therefore Mm' = 3c$.

Hence, taking the middle point, O, of the horizontal portion, RQ, as origin, and the horizontal and vertical lines through it as axes of x and y, the co-ordinates of P are $(\frac{3}{2}a,\ c)$; those of N are $(\frac{5}{2}a,\ c+2c)$;

those of *M* are $(\frac{7}{2}a,\ c+2c+3c)$; and those of the n^{th} vertex from *Q* are evidently

$$x=\frac{2n+1}{2}\cdot a, \qquad y=\frac{n(n+1)}{2}\cdot c.$$

The value of the ordinate, y, of any vertex at once enables us to determine this vertex.

If we eliminate n from the two equations for x and y, we get an equation which is satisfied by all the vertices indifferently. This equation denotes, therefore, a curve passing through all the vertices of the polygon. Eliminating n, we get

$$x^2=\frac{2a^2}{c}\cdot y+\frac{a^2}{4}.$$

This denotes a parabola whose axis is the vertical line Oy. The vertex of the parabola is vertically below O at a distance $=\frac{c}{8}$.

The smaller the distances RQ, Qp, pm, . . . , the more nearly does the Funicular Polygon coincide with the parabolic curve.

10. To represent graphically the forces in the general case of the Funicular Polygon.

For convenience, let the vertices of the string or chain be denoted by the numbers 1, 2, 3, . . . , and let the forces P_2, P_3, . . . act at the vertices. Let also the tension in the portion of the string (1, 2) be denoted by T_{12}, &c.

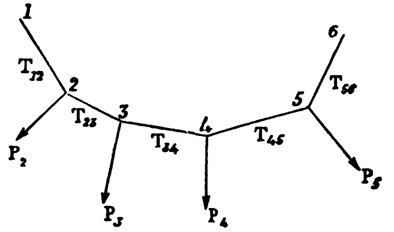

Fig. 29.

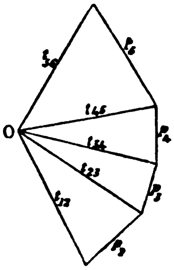

Fig. 30.

Now, take any point, O, and from it draw the line t_{56} parallel to the string (5, 6), and proportional to the tension T_{56}. From the extremity of t_{56} draw the line, p_5, parallel and proportional to the force P_5. It follows, then, that since the forces T_{56}, T_{45}, and P_5 form a system in equilibrium at the point (5), the third side, t_{45}, of the triangle t_{56}, p_5, t_{45} is parallel to T_{45}, and proportional to it (Cor. 2, p. 23). In the same way, drawing p_4 parallel and proportional to P_4, the side t_{34} is parallel and proportional to T_{34}; and continuing this construction, the tensions in the successive portions of the string are all represented by the lines t_{12}, t_{23}, t_{34}, . . . in the new figure (Fig. 30).

The figure (Fig. 30) which represents by its lines, both in magnitudes and in directions, all the forces of the system in Fig. 29, has been called by Clerk Maxwell a 'Force Diagram' of the system. (Transactions of the Royal Society of Edinburgh, vol. xxvi.)

When, as in example 8, all the applied forces, P_2, P_3, . . . are parallel, the Force Diagram of the system consists of a triangle with lines drawn from the vertex to different points in the base. Thus, taking any point, O (Fig. 31), and drawing Ob parallel to MB (Fig. 28), and proportional to the tension in it; and then drawing bm vertical and proportional to the weight of the mass suspended at M, it follows that Om will be parallel to MN, and proportional to the tension in it. Similarly for the rest of the figure. If all the weights are equal, the lines bm, mn, np, pq, . . . are all equal, and Fig. 31 at once shows that the tangents of the successive inclinations of the parts of the chain are in Arithmetical Progression. This figure also exhibits the constancy of the horizontal components of the tensions Ob, Om, On, ... these components being all represented by Oq.

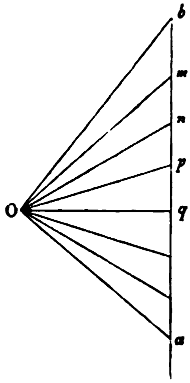

Fig. 31.

11. Masses of 20, 14, 20, 16, 10, and 18 kilogrammes are to be suspended from the vertices of a funicular polygon; and these vertices are to be on vertical lines which are, respectively, 3, 2, 2½, 2½, and 4 decimètres apart; find the figure of the polygon, and also, by scale measurement, the tensions in the different portions of the string so that each one, successively, of the following conditions is satisfied:—

(α) The portion of the string between the third and fourth lines to be horizontal, and the portion which is attached to the 20 kilogrammes and to a fixed point to make an angle of $\frac{\pi}{6}$ with the vertical.

(β) The extreme portions of the string (those attached to fixed points) to make each an angle $\frac{\pi}{3}$ with the vertical.

(γ) The tensions in the portions between the second and third lines and between the fourth and fifth lines to be each equal to twice the tension in the portion between the third and fourth lines.

(δ) The extreme tensions to be 60 and 70 kilogrammes' weight, respectively.

(ϵ) The vertices on the first and sixth lines to be in the same horizontal line, and the vertex on the fourth line to be 5 decimètres below that on the first line.

Let Fig. 27 represent the polygon, the given vertical lines being those marked MW, NW, PW, ... and the fixed ends of the string being B and A. Draw a vertical line; let a_0 be the upper end of this line; measure off lengths $a_0 a_1$, $a_1 a_2$, $a_2 a_3$, $a_3 a_4$, $a_4 a_5$, $a_5 a_6$ respectively proportional to 20, 14, 20, 16, 10, 18. Then the solution consists, in each case, in finding a proper position for the point O, which is called the *pole* of the funicular polygon.

Now in (α) the line Oa_3 is to be horizontal, so that O must be somewhere on the horizontal line through a_3. Also since the line BM makes an angle $\frac{\pi}{6}$ with the vertical, the line Oa_0 must make this angle with the line $a_0 a_6$. Hence O must lie on another given line; and O is therefore determined. In every case the actual position of the first vertex, M, is, of course, arbitrary, so that we may assume B but not A, or *vice versâ*.

Case (β) is similarly solved. In (γ), since the tensions referred to are represented on the scale adopted by the lengths Oa_2, Oa_4, Oa_3, we must have $Oa_2 = Oa_4 = 2\,Oa_3$. Now since $Oa_2 = Oa_4$, the point O must lie on the line bisecting $a_2 a_4$ perpendicularly; and since $Oa_2 = 2\,Oa_3$, the pole O must lie on a circle having for diameter the line joining the points which divide the line $a_2\ a_3$ internally and externally in the ratio 2 : 1. Hence O is determined.

In (δ), the lengths Oa_0 and Oa_6 are given; $\therefore$ O is determined. To solve (ϵ), let $\theta_{01}, \theta_{12}, \ldots . \theta_{67}$ be the inclinations (all measured in the same sense) of the portions BM, MN, . . . to the vertical; so that $Oa_0a_1 = \theta_{01}$, $Oa_1a_2 = \theta_{12}$, . . . $Oa_5a_6 = \theta_{56}$. Then, taking an origin anywhere on the vertical line MW, if y_1, y_2, y_3, . . . are the vertical distances in decimètres of the successive vertices below this origin, we have

$$y_2 - y_1 = 3\cot\theta_{12};\ y_3 - y_2 = 2\cot\theta_{23};\ y_4 - y_3 = 2\tfrac{1}{2}\cot\theta_{34};$$
$$y_5 - y_4 = 2\tfrac{1}{2}\cot\theta_{45};\ y_6 - y_5 = 4\cot\theta_{56}. \qquad (1)$$

Add these, and observe that y_6 is given equal to y_1. Hence

$$3\cot\theta_{12} + 2\cot\theta_{23} + 2\tfrac{1}{2}\cot\theta_{34} + 2\tfrac{1}{2}\cot\theta_{45} + 4\cot\theta_{56} = 0. \qquad (2)$$

Now observing that a_1 divides the line $a_0\,a_6$ so that $\frac{a_0 a_1}{a_1 a_6} = \frac{20}{78}$, we have (using θ and ϕ instead of θ_{01} and $\pi - \theta_{67}$), from Art. 36, $\cot\theta_{12} = \frac{1}{98}(78\cot\theta - 20\cot\phi)$; and similarly $\cot\theta_{23} = \frac{1}{98}(64\cot\theta - 34\cot\phi)$; &c. Substituting these in (2) we have

$$307\cot\theta - 379\cot\phi = 0, \qquad (3)$$

which shows that O must lie on a horizontal line dividing $a_0 a_6$ in the ratio 379 : 307.

To express the second condition, we have from (1)

$$y_4 - y_1 = 3\cot\theta_{12} + 2\cot\theta_{23} + 2\tfrac{1}{2}\cot\theta_{34} = 5;$$

which becomes $\quad 472\cot\theta - 263\cot\phi = 490, \qquad (4)$

and this shows (Art. 36) that O must lie on a right line dividing $a_0 a_6$ in the ratio 263 : 472 and making with it the angle whose cotangent $= \frac{98}{147}$, i. e., about 56° 19′.

The two loci (3) and (4) determine O completely.

12. For any given system of vertical lines and weights, show how to construct a funicular polygon such that two assigned sides of it shall pass each through a given point.

Ans. Let the given weights at the vertices be w_1, w_2, w_3, ... and the horizontal distances between the given lines c_{12}, c_{23}, c_{34}, ... Let any two sides—say that between lines 2 and 3, and that between lines 5 and 6—pass through two points whose distances from any horizontal line are β and β', and whose distances from the lines 3 and 5, respectively, are p and p'. Then with the notation of last example, we have

$$y_3-\beta = p\cot\theta_{23};\ y_4-y_3 = c_{34}\cot\theta_{34};\ y_5-y_4 = c_{45}\cot\theta_{45};$$
$$\beta'-y_5 = p'\cot\theta_{56};$$
$$\therefore\quad \beta'-\beta = p\cot\theta_{23}+c_{34}\cot\theta_{34}+c_{45}\cot\theta_{45}+p'\cot\theta_{56}. \qquad (a)$$

And as in last example, we have (Art. 36)

$$(w_1+w_2+w_3+\ldots)\cot\theta_{23} = (w_3+w_4+\ldots)\cot\theta-(w_1+w_2)\cot\phi,$$

with similar values of $\cot\theta_{34}$, ... so that (a) becomes of the form

$$l\cot\theta-m\cot\phi = \beta'-\beta,$$

where l and m are given. This shows that the locus of O is a right line, which can be easily drawn. Any funicular, therefore, constructed from a pole on this line satisfies the given conditions.

13. For any given system of vertical lines and weights, show how to construct a funicular such that three assigned sides of it shall pass each through a given point.

14. **Suspension Bridge.** The number of vertices of the polygon being very great, and the suspended masses all equal, the parabola which passes through all the vertices virtually coincides with the chain forming the polygon, and gives the figure of the Suspension Bridge. In this bridge the masses, W, W,..., suspended from the successive portions of the chain are those of equal portions of the flooring. The weight of the chain itself and the weights of the sustaining bars are negligible in comparison with the weight of the flooring and the load which it carries.

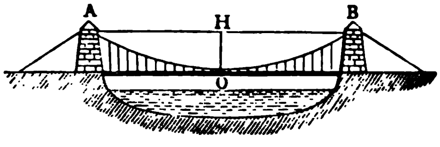

Fig. 32.

Fig. 31 may be taken to represent the Force Diagram of the Suspension Bridge, the vertical line ab, representing the weight of the flooring, being divided into as many equal parts as there are divisions of the chain. If these parts are sufficiently numerous, the lines Ob, Om, On, &c. are parallel to tangents to successive points of the chain. Let the span, AB, of the bridge $= 2a$, and let the height $OH = h$. Then, the equation of the parabola referred to horizontal and vertical axes of y and x, respectively, through O (Fig. 32) is

$$y^2 = 4mx,$$

m being a constant; and the tangent of the inclination to the vertical of any portion

$$= \frac{dy}{dx} = \frac{2m}{y} = \frac{y}{2x}.$$

Hence the tangent at the point of support, B, makes with the horizon an angle whose tangent is $\frac{2h}{a}$.

Therefore, Oq (Fig. 31) being parallel to the tangent at the lowest point of the bridge, and Ob parallel to the tangent at the point B,

$$\tan bOq = \frac{2h}{a}.$$

Hence, since bq represents half the weight of the bridge, and Ob the terminal tension of the chain at B,

$$\text{Terminal tension} = \frac{W}{2\sin bOq} = W\frac{\sqrt{a^2+4h^2}}{4h},$$

W being the weight of the flooring.

Also, the vertical tension at $B = \frac{1}{2}W$, and the constant

$$\text{Horizontal tension} = W\frac{a}{4h}.$$

15. The entire load of a suspension bridge is 160,000 kilogrammes, the span is 64 mètres, and the height is 5 mètres; find the tension at the points of support, and also the tension at the lowest point.

Ans. Terminal tension = 268,208 kilogrammes' weight.
Horizontal tension = 256,000 " "

16. If the vertical bars which support the roadway of a suspension bridge are not at equal horizontal distances, prove that the vertices of the polygon formed by the chain will still lie on a parabola, provided that each vertical bar supports half of the adjacent portions of the roadway.

This follows from the fact that the cotangent of the inclination of any chord of a parabola to the axis is proportional to the sum of the ordinates of the extremities of the chord.

17. If R is the resultant of any number of forces, P_1, P_2, P_3,......, acting in one plane on a particle, prove that

$$R^2 = \Sigma P^2 + 2\Sigma P_1 P_2 \cos(\widehat{P_1, P_2}),$$

where $\widehat{P_1, P_2}$ means the angle between P_1 and P_2.

(This result is true for non-coplanar forces.)

18. If a particle is in equilibrium under the action of any forces, prove that the sum of the *oblique* components of the forces along any right line is zero.

If ΣX and ΣY denote the sums of the components along two lines inclined at an angle $= \omega$, the square of the resultant is equal to

$$(\Sigma X)^2 + 2(\Sigma X)(\Sigma Y)\cos\omega + (\Sigma Y)^2;$$

and this $\equiv (\Sigma X + \Sigma Y)^2 \cos^2\frac{\omega}{2} + (\Sigma X - \Sigma Y)^2 \sin^2\frac{\omega}{2}.$

Hence the result follows as in equations (3), p. 30. It is otherwise evident, since the resultant is the third side of a triangle, two of whose sides are ΣX and ΣY, so that both sides must vanish.

19. If in example 7 the masses W and W', instead of being knotted to two given points in the string, are attached to two smooth rings which are capable of sliding freely along the string, determine the condition and position of equilibrium.

Here, since the string passes freely over and under smooth surfaces, the tension is constant throughout its length. Now, the tension in Am is P, and that in $Cm' = P'$. Hence

$$P = P'.$$

For the equilibrium of m, we have, resolving vertically,

$$W = 2P\sin\theta; \ \therefore \sin\theta = \frac{W}{2P};$$

and for the equilibrium of m',

$$W' = 2P\sin\theta'; \ \therefore \sin\theta' = \frac{W'}{2P}.$$

20. A and B are two fixed pegs; over A passes a string MAC, and over B a string NBC, knotted each at C to a mass of weight W, and sustaining at their free ends, M and N, masses of weights P and Q, respectively; find the position of equilibrium of the system.

Ans. The inclinations of AC and BC to the vertical are at once given by the triangle of forces.

21. A heavy particle is attached to one end of a string, the other end of which is fixed. Find the horizontal force which must be applied to the particle in order that the string may deviate by a given angle from the vertical, and find also the tension of the string.

Ans. If $F =$ the horizontal force required, $T =$ tension of string, $W =$ weight of particle, and $\theta =$ angle of string's deviation,

$$F = W\tan\theta, \ T = W\sec\theta.$$

22. A string ACB (Fig. 25, example 5) has its extremities tied to two fixed points, A and B; to a given point, C, in the string is knotted a given mass, W. Find the tensions in the portions CA and CB.

Ans. Since AC and BC are given, the angles CAB and CBA are also given. If these angles are denoted by θ and θ', and if T and T' are the tensions in CA and CB,

$$T = \frac{W\cos\theta'}{\sin(\theta+\theta')}, \ T' = \frac{W\cos\theta}{\sin(\theta+\theta')}.$$

23. If (same figure) the extremities A and B are fixed, and the

weight W is that of a smooth, heavy ring at C, which is capable of sliding freely along the string, find the horizontal force which must be applied to the ring C in order that the system may take a given position of equilibrium.

Ans. If the angles CAB and CBA are θ and θ', and $F =$ the required force,

$$F = W \tan \frac{\theta - \theta'}{2}.$$

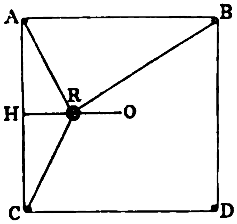

Fig. 33.

24. $ABCD$ (Fig. 33) is a system of pegs forming a square in a vertical plane; a string attached to A and B passes through a heavy, smooth ring, R, while another string is attached to C and R. The ring is kept in equilibrium half way between H, the middle point of CA, and O, the centre of the square; find the tensions in the strings ARB and CR.

Ans. If $W =$ weight of ring, $T =$ tension in ARB, and $T' =$ tension in CR,

$$T = W . \frac{13\sqrt{5} + 5\sqrt{13}}{32}, \quad T' = W . \frac{\sqrt{5} + 5\sqrt{13}}{16}.$$

25. In the last example if the tensions in the two strings are equal, find the point at which the ring must be placed on OH.

Ans. If $\frac{OR}{OH} = x$, x is determined by the equation

$$3x^4 - 3x^2 - 10x + 6 = 0.$$

This equation has only two real roots, one between 0 and 1, and the other between 1 and 2.

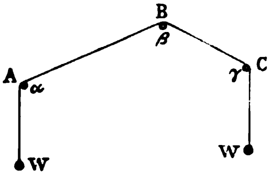

Fig. 34.

26. A string whose weight is neglected passes over three smooth pegs, A, B, C (Fig. 34), in a vertical plane, and sustains two equal masses, W, from its extremities. Find the pressures on the pegs; and find also the magnitudes of the angles α, β, and γ, when the system of pegs is least likely to break, the pegs being all equally strong.

Ans. If P, Q, and R be the pressures on the pegs A, B, and C, respectively, $P = 2W \cos \frac{\alpha}{2}$, $Q = 2W \cos \frac{\beta}{2}$, $R = 2W \cos \frac{\gamma}{2}$; and since the sum of α, β, and γ is given $(= 2\pi)$, it follows that in the best arrangement $\alpha = \beta = \gamma = \frac{2}{3}\pi$. For, unless each of the angles $= \frac{2}{3}\pi$, some one of the pressures must be $> 2W \cos \frac{\pi}{3}$, or W; and if the pegs are of equal strength, i. e., have all the same breaking pressure, and the suspended weight W is gradually increased, the breaking pressure on the peg at which the angle between the two portions of the string is least will be reached for a less value of W than if each angle $= \frac{2}{3}\pi$.

27. The ends of a string are attached to two fixed points, A, B, in the same horizontal line, at given points, C, D, in the string are fastened two masses, P and Q; find the relation which must hold between the given magnitudes so that the portion CD of the string may in the position of rest be horizontal.

Ans. If $AB = a$; $BC = l$ (C being adjacent to B, and D to A); $CD = m$; $DA = n$; then

$$P(\overline{a-m}^2 + l^2 - n^2) = Q(\overline{a-m}^2 - l^2 + n^2).$$

28. In the last if the portion CD is to make a given angle, i, with the horizon, find the ratio $P : Q$.

Ans. $P(\tan\phi + \tan i) = Q(\tan\theta - \tan i)$, where θ and ϕ are the inclinations of BC and AD to the horizon, these angles being thus determined: from B draw BE at the lower side of BA, equal to m and making $ABE = i$; on AE construct the triangle ADE whose sides AD, DE are n, l, respectively; hence the quadrilateral $BCDA$ is completely determined.

29. Three smooth pegs, A, B, C, are fixed in a vertical plane; three light strings knotted together at a common extremity, O, have suspended from their other extremities given masses, P, Q, R, and the corresponding strings are passed over the pegs A, B, and C; find the position of equilibrium.

Ans. Construct a triangle whose sides are proportional to the magnitudes P, Q, R; then the external angles of this triangle are equal to the angles BOC, COA, AOB; so that the point O is determined as the intersection of circular arcs described on BC and CA.

30. If in the last example the knot O is replaced by a smooth ring of negligible weight which is tied to the peg C by a string of given length, while another string, passing freely through the ring, passes over the pegs A and B, and has two given masses suspended from its extremities; find the position of equilibrium.

Ans. [The suspended masses must be equal.] Describe an ellipse having A and B for foci and touching the circle described with C as centre and CO as radius. See ex. 40, and graphics in Chapter III.

31. If the string in ex. 25 passes over any number of equally strong, smooth pegs, in the same vertical plane, find the best arrangement.

Ans. If there are n pegs, each of the angles, α, β, γ, δ, . . . must be $= \frac{(n-1)\pi}{n}$.

32. In example 19 calculate the pressures on the pegs A, B, C.

Ans. The squares of the pressures are respectively

$$P(2P+W),\ \tfrac{1}{2}\{4P^2 + WW' - \sqrt{(4P^2 - W^2)(4P^2 - W'^2)}\},\ P(2P+W').$$

33. If the strengths of the pegs, A, B, C, in example 25, are proportional to l, m, n, find the best arrangement of the system.

Ans. The angle a is given by the equation

$$2mnx^3+(l^2+m^2+n^2)x^2-l^2=0,$$

in which $x=\cos\frac{a}{2}$. The angles β and γ are at once found from a.

34. Prove that the unknown angles in ex. 7, p. 39 may be found by the following construction: draw a right line OA proportional to W; produce this line through A to O' so that AO' represents W'; on OO' produced take a point Q so that $QO:QO'=a.W':a'.W$; round O and O' describe circles whose radii represent P and P', respectively; on the line QO take a point, L, between Q and O such that $QL=kWW'/(a\cdot W'-a'\cdot W)$; and at L erect the perpendicular LM to QL. Take a point X on the circle round O, and a point Y on the circle round O' such that $AX=AY$; let OX and $O'Y$ meet in Z; draw ZQ meeting LM in R. Then if X is so chosen that $LR=AX=AY$, the angles XOA, XAO, YAO', $YO'A$ are the complements of $\theta, \phi, \phi', \theta'$, and the tension of the part of the string between the knots is represented by AX.

35. Let $A_0 A_1 \ldots A_5$ (Fig. 35) be any funicular polygon, with masses whose weights are P_1, P_2, P_3, P_4 suspended at its vertices A_1, A_2, A_3, A_4, respectively; draw any line, $a_0\, a_5$, meeting the verticals through $A_0, A_1, \ldots$ in the points $a_0, d_1, d_2, \ldots$ and let $A_0 A_5$ meet these verticals in $A_0, D_1, D_2, \ldots$ Now construct a new polygon, $a_0 a_1 a_2 \ldots a_5$, by taking $d_1 a_1 = \frac{1}{n} D_1 A_1$; $d_2 a_2 = \frac{1}{n} D_2 A_2$; and so on, n being any number.

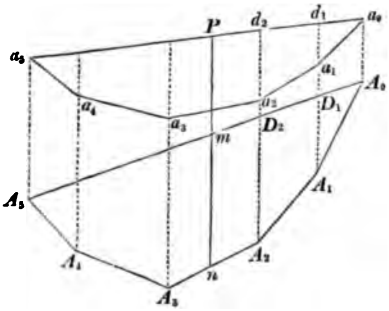

Fig. 35.

Prove that the new polygon, whose fixed ends are a_0 and a_5, will be kept in equilibrium by the set of forces P_1, P_2, P_3, P_4 applied at its vertices a_1, a_2, a_3, a_4.

Although this may be readily proved geometrically by principles of Graphic Statics, the student will do well to establish it by the method of example 8. He will easily prove that, if a and β are the inclinations of $A_0 A_5$ and $a_0 a_5$ to the horizon, $\theta_{01}, \theta_{02}, \ldots$ the inclinations of the sides $A_0 A_1$, $A_1 A_2, \ldots$, and $\phi_{01}, \phi_{12}, \ldots$ those of $a_0 a_1$, $a_1 a_2, \ldots$ to the horizon, we shall have

$$\tan\phi_{01}-\tan\beta=\frac{1}{n}(\tan\theta_{01}-\tan a);$$

$$\tan\phi_{12}-\tan\beta=\frac{1}{n}(\tan\theta_{12}-\tan a), \text{ \&c.}$$

But if T denotes the constant horizontal tension in a funicular polygon, the conditions of its equilibrium are

$$\tan\theta_{01}-\tan\theta_{12}=\frac{P_1}{T};\quad \tan\theta_{12}-\tan\theta_{23}=\frac{P_2}{T};\quad \&c.$$

These conditions are satisfied in the polygon $a_0\,a_1,\ldots a_5$ on the supposition that the horizontal component of tension in it $= n\,T$; and it is axiomatic that if internal forces can preserve equilibrium, they will.

Of course all the ordinates (and not merely those through the vertices) of the derived polygon are proportional to the corresponding ordinates of the original.

36. Show that the last example enables us to construct for a given parallel system of forces a funicular polygon which shall pass through three given points.

(A solution of this problem for any system of forces will be given in a subsequent chapter.)

37. Given the base, NS (Fig. 36), of a triangle NPS, and also the sum of the cosines of the base angles, SNP and NSP; let the curve locus of P be constructed. Prove that if a particle be placed at any point of the curve and acted on by two forces, one repulsive from N and equal to $\frac{\mu}{NP^2}$, and the other attractive towards S and equal to $\frac{\mu}{SP^2}$, the resultant force is, at every position of the particle, directed along the tangent to the curve.

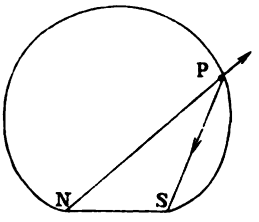

Fig. 36.

N.B.—This curve is called the 'Magnetic Curve,' being one of those in which small iron filings would arrange themselves under the influence of a fixed magnet whose poles are N and S.

It is to be observed that each little piece of iron is a magnet, having two poles at its extremities, and that it must therefore set at the point, P, where it is placed, in the direction of the resultant force on either of its poles.

38. Prove that the line of action of the resultant force of a magnet on a magnetic pole at P divides NS externally in the ratio $NP^3 : SP^3$.

39. Iron filings are sprinkled over a sheet of paper on which a magnet rests; prove that all those filings which dip towards the same point on the line of the magnet lie on a circle (neglecting their mutual actions).

40. In ex. 30 show that the position of the ring O may be thus found: let $CD\,(=d)$ be the diagonal of the parallelogram whose sides are CA, CB; let $CA=b$, $CB=a$, $CO=r$, $\angle OCD=\phi$, $\epsilon=\angle$ between CD and bisector of $\angle C$; then

$$ab\sin 2(\phi-\epsilon)=rd\sin\phi.$$

CHAPTER III.

THE EQUILIBRIUM OF A PARTICLE ON PLANE CURVES.

Section I.

Smooth Curves.

37.] **Smooth surface.** When a body is placed in contact with a surface, it is evident that, in addition to the given forces acting on the body, there is a certain force produced by the surface—the force, namely, which the surface exerts to prevent the body from passing through it. This force is called the *Reaction* of the surface. Now, the surface being supposed to be rigid, there is evidently no limit to the *magnitude* of the force with which it is capable of reacting; but the *direction* of the force depends on the nature of the surface itself. If the surface be perfectly smooth, it can react on any body in contact with it only in the direction of the *normal* to the surface at the point where the body is in contact with it. Thus (Fig. 37), if a body, M, acted on by any given system of forces, be in contact at a point O with a *smooth* surface, AB, the force which this surface exerts on the body takes the direction, ON, of the normal to the surface at the point of contact, O, and its magnitude will be such as to destroy the effect of all the other forces acting upon M. To the magnitude of the reaction, R, there is no limit; so that if each of the other forces acting on M were increased 100 times, for example, the surface would react with a force equal to 100 R; but the *direction* of R is strictly limited to that of the normal. We may, then, define smooth surfaces thus—

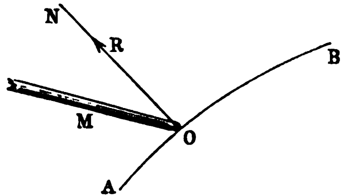

Fig. 37.

When two smooth bodies are in contact, their mutual reaction is normal to the surface of contact.

38.] **Example.** If P (Fig. 38) is a heavy particle whose weight is W, placed on a smooth spherical surface whose vertical diameter is AB, what is the position of equilibrium?

Here the forces acting on P are only two in number—namely, its weight, W, and R, the reaction of the smooth surface. Now, this reaction takes place in the direction of the normal, PO, to the sphere at P; and since the particle is in equilibrium under the action of only two forces, these must be equal in magnitude, and act in opposite directions. Hence, since W acts vertically, PO must be a vertical line; that is, P must be placed at A, the lowest point of the sphere, or outside the surface at B, the highest point.

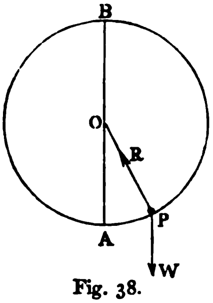

Fig. 38.

Whatever be the smooth surface on which the particle is placed, it is evident that the points on it at which the particle will rest, under the influence of its weight and the reaction alone, are points the normals at which are vertical lines. And, generally—

A particle will rest at those points of a smooth surface at which the normal coincides with the direction of the resultant of all the forces acting on the particle.

39.] **Normal to a Curve.** The normal to a curve at a given point is not, like the normal to a surface at a given point, a definite line, but is *any line whatever in the plane perpendicular to the tangent at the point.*

Hence, for the equilibrium of a particle placed inside a smooth tube of any form, the resultant force on the particle need not act in a given right line, but must act in a given plane—namely, the plane which is normal to the tube at the point where the particle is placed. Thus, for example, let AB (Fig. 39) be a smooth tube of any form, and let P be a particle placed inside it. If we imagine a string attached to P, coming out of the tube through an opening at P, which is not sufficiently large to allow P to come out, it is evident that we may pull at P with any force however great

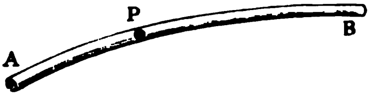

Fig. 39.

in the plane normal to the tube, and in all directions round P and the equilibrium of the particle will not be disturbed. But if we incline the string ever so little to the normal plane at P, motion will ensue along the tube.

40.] **Plane Curve.** In the present chapter we shall consider only *plane curves*, i.e., curves which lie altogether in one plane.

Moreover, when a particle is placed on a curve, and acted on by given forces, we shall suppose that all the forces act in the plane of the curve.

Now, it is evident that the only effect which a curve produces on a particle placed upon it is a normal reaction of some definite magnitude. If, then, we produce upon the particle, by any other means, a force identical with this reaction, we may dispense with the curve altogether. This being so, if we call the reaction of the curve R, we may suppose the particle acted upon by all the given forces, and also by a new force equal to R, this latter acting in the direction of the normal to the curve. Thus, the case is the same as that treated in the last chapter—namely, the equilibrium of a particle acted upon by any number of forces in one plane; and in writing down the equations of equilibrium, we shall merely have to include the new force R among all the others.

41.] **Graphic Solution of Equations.** It often happens that a position of equilibrium is defined by two angles for which two equations are given. The equation for either variable which results from eliminating the other may be one of high degree, the approximate solution of which by the methods of the Theory of Equations would be very troublesome. In such cases it is often possible to obtain a solution sufficiently accurate for practical purposes by constructing curves corresponding to the equations and taking their points of intersection. For this purpose a box of mathematical instruments is required.

A few illustrations of the method, as well as some examples of frequent occurrence, are here given, but in numerous instances of like character, which will present themselves subsequently, the student must exercise his ingenuity in obtaining graphic solutions for himself.

(a) If A and B are two fixed points and P a variable point, whose position is defined by the angles PAB $(=\theta)$ and PBA

($= \phi$), what locus is represented by the equation

$$a \cot(\theta - \alpha) + b \cot(\phi - \beta) = c,$$

where a, b, c, α, β are constants?

It will be easily found that it denotes, in general, a conic circumscribing the triangle ABC, where C is determined by drawing AC making $\angle CAB = \alpha$, and BC making $\angle CBA = \beta$.

But if $a \cot\alpha + b \cot\beta + c = 0$, it will be found that the equation represents a right line, the conic becoming the product of this line and the line AB itself.

In particular, $a \cot\theta + b \cot\phi = c$ denotes a right line, which is constructed by producing AB to D, so that $AB : BD = b - a : a$; and at D drawing the line DC, making the angle

$$CDB = \cot^{-1} \frac{c}{b-a}.$$

(β) With the same meanings of θ and ϕ, construct the locus represented by the equation

$$a \cos\theta + b \cos\phi = c.$$

With the points A and B as centres describe two circles, S and T (Fig. 40) of radii $\frac{a}{c} \cdot AB$ and $\frac{b}{c} \cdot AB$, respectively. Draw any common ordinate NLQ meeting them in L and N; then the lines AL and BN intersect in a point, P, on the required locus; for

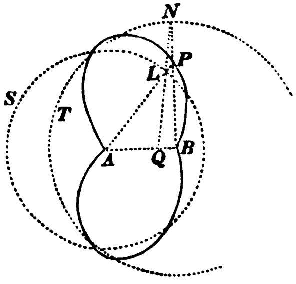

Fig. 40.

$$AQ + QB = AB, \text{ or}$$

$$AL \cdot \cos\theta + BN \cdot \cos\phi = AB, \text{ or}$$

$$\frac{a}{c} \cdot AB \cos\theta + \frac{b}{c} \cdot AB \cos\phi = AB,$$

which is the given equation.

By drawing an indefinite number of lines, such as NQ, perpendicular to AB to cut both circles, we determine as many points as we choose on the curve, which is represented in the figure by the thick line.

In the particular case in which $a = b$ the locus is the *Magnetic Curve* (p. 51).

(γ) To find θ and ϕ from the equations

$$\frac{a}{\sin\theta}+\frac{b}{\sin\phi}=c \text{ and } \cos\theta=k\cos\phi,$$

where a, b, c, k are constants.

Take two points, A and B (Fig. 41), such that $AB=a+b$; take $AO=a$, $OB=b$, and draw OD perpendicular to AB; with A as centre and c as radius describe a circle; draw any radius, AC, of this circle, meeting OD in L; inflect BJ equal to LC; then P, the point of intersection of AC and BJ, is a point on the locus, the angles θ and ϕ being ALO and BJO, respectively. The full curve represents half the locus, there being a similar portion below AB.

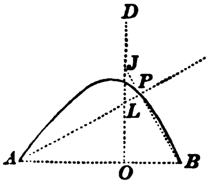

Fig. 41.

Also the equation $\cos\theta=k\cos\phi$ gives $\sin PAB=k.\sin PBA$, or $PB=k.PA$, so that the locus representing this equation is a circle whose diameter is the line joining the points which divide AB internally and externally in the ratio $1:k$. The values of θ and ϕ which satisfy both equations are those belonging to the points of intersection of this circle and the previous curve.

(δ) To find θ from the equation

$$a\sin\theta+b\cos\theta=c.$$

We may, of course, form a quadratic for either $\sin\theta$ or $\cos\theta$, but when a, b, c are numerically given, this method would often be very troublesome.

Divide out by a, and put $\frac{b}{a}=\tan\alpha$; multiply by $\cos\alpha$, and we get

$$\sin(\theta+\alpha)=\frac{c}{a}\cos\alpha.$$

The angle α is known from a table of natural tangents, so that this last equation gives θ at once.

As a numerical example, let it be required to find the inclination of a smooth plane on which a mass of 7·5 kilogrammes can be sustained by an up-plane force of 2·4, and a horizontal force of 3·6 kilogrammes' weight. The equation for i, the inclination, is

$$7{\cdot}5\sin i-3{\cdot}6\cos i=2{\cdot}4.$$

Dividing out by 7·5, and looking in the tables for $\tan^{-1}\frac{3{\cdot}6}{7{\cdot}5}$, we have

$$\sin i-\cos i\tan(25^\circ\ 38'\ 28'')={\cdot}32.$$

Multiplying by cos (25° 38′ 28″), we have

$$\sin(i - 25^\circ\ 38'\ 28'') = \cdot 2884871 = \sin(16^\circ\ 46'\ 3''),$$

$$\therefore\ i = 42^\circ\ 24'\ 31''.$$

Examples.

1. A heavy particle is placed on a smooth inclined plane, AB (Fig. 42), and is sustained by a force, F, which acts along AB in the vertical plane which is at right angles to AB; find F, and also the pressure on the inclined plane.

The only effect of the inclined plane is to produce a normal reaction, R, on the particle. Hence, if we introduce this force, we may imagine the plane removed. Let W be the weight of the particle, and i the inclination of the plane to the horizon.

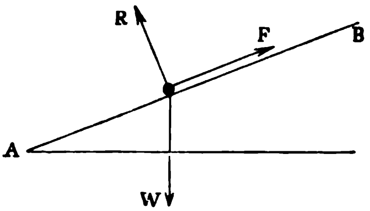

Fig. 42.

Resolving the forces along AB, we have

$$F - W\sin i = 0, \text{ or } F = W\sin i;$$

and, resolving perpendicularly to AB,

$$R - W\cos i = 0, \text{ or } R = W\cos i.$$

If, for example, the mass of the particle is 4 grammes and the inclination of the plane 30°, there will be a normal pressure of $2\sqrt{3}$ grammes' weight on the plane, and the force F will be 2 grammes' weight.

2. In the previous example, if F act horizontally, find its magnitude, and also that of R.

Resolving along AB, and perpendicularly to it, we have, successively,

$$F\cos i - W\sin i = 0, \text{ or } F = W\tan i;$$

and $$F\sin i + W\cos i - R = 0, \ \therefore R = \frac{W}{\cos i}.$$

R is therefore in this case greater than it was before, as is sufficiently evident *a priori*.

3. If the particle is sustained by a force, F, making a given angle, θ, with the inclined plane, find the magnitude of this force, and of the pressure, all the forces acting in the same vertical plane.

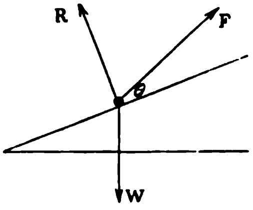

Fig. 43.

Resolving along the plane (Fig. 43),

$$F\cos\theta - W\sin i = 0, \text{ or } F = \frac{W\sin i}{\cos\theta};$$

and resolving perpendicularly to the plane,

$$R + F\sin\theta - W\cos i = 0, \ \therefore R = W\frac{\cos(i+\theta)}{\cos\theta}.$$

The student will, of course, observe that these values of F and R could have been at once obtained, without resolution, by the equation (a), p. 15.

4. A heavy particle, whose weight is W, is sustained on a smooth inclined plane, by three forces applied to it, each equal to $\frac{W}{3}$; one acts vertically, another horizontally, and the third along the plane (Fig. 44); find the inclination of the plane.

Since we do not require the value of R, the pressure on the plane, we shall resolve forces at right angles to R, that is, along the plane. Hence

$$\frac{W}{3}\sin i + \frac{W}{3} + \frac{W}{3}\cos i - W\sin i = 0,$$

or
$$2\sin i = 1 + \cos i, \ \therefore\ 2\sin\frac{i}{2}\cos\frac{i}{2} = \cos^2\frac{i}{2}. \qquad (1)$$

If we reject the factor $\cos\frac{i}{2}$, for the present, we have

$$\tan\frac{i}{2} = \frac{1}{2},$$

which determines the inclination.

Now the expulsion of the factor $\cos\frac{i}{2}$ from equation (1) amounts to rejecting the solution

$$\cos\frac{i}{2} = 0.$$

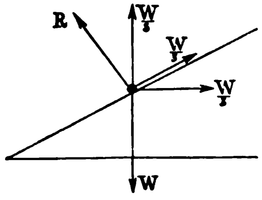

Fig. 44.

But in this, as well as in many physical and geometrical problems, such a solution ought not to be rejected, unless it is shown to be irrelevant to the question. So long as our equations are *perfect* interpretations of the physical or geometrical conditions of the problem, no factor can furnish an irrelevant solution. It is only when an equation expresses more or less than is implied in the given conditions that irrelevant factors can present themselves. Instances of these factors frequently occur in the operations of Algebra and Analytic Geometry—as, for example, when we rationalize an equation by the process of squaring. If, before this process, the square root of a quantity was affected with a *minus* sign, this sign will be indifferent in the rationalized result, and this latter, consequently, expresses *more* than was contained in the original equation. Hence it may happen that the result will furnish us not only with what is relevant, but, in addition, with what is wholly irrelevant.

In the present instance the equation $\cos\frac{i}{2} = 0$ would give the inclination of the plane $= 180°$, and the figure would then become Fig.

45, in which the particle is placed underneath the plane in such a way that equilibrium is manifestly impossible.

Hence it appears as if the equation $\cos\frac{i}{2} = 0$ were wholly without meaning.

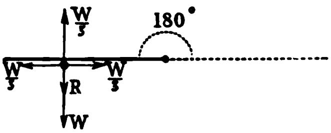

Fig. 45.

A little reflection, however, will show that it is quite relevant. For equation (1) is merely the analytical expression of the physical condition that the component of the acting forces *along the plane* shall be zero, the reaction of the plane being presumed to assume such a value as to render the total component of force equal to zero in the normal direction. Now it is not enough *for equilibrium* that the component along some one line shall be zero; for this, the component along some other line must vanish as well. Hence our result does not express the complete condition of the particle's equilibrium, but merely *a part* of that condition; and each of the equations

$$\tan\frac{i}{2} = \frac{1}{2}, \text{ and } \cos\frac{i}{2} = 0,$$

expresses perfectly all the physical conditions contained in (1). For when the inclination is 180°, the force $\frac{W}{3}$ which acted along the inclined plane becomes a horizontal force opposite to the given horizontal force $\frac{W}{3}$; and the vertical $\frac{W}{3}$ furnishes no component along the plane. But in this position the normal force should consist of a *pull*, i.e., either the particle should be placed on the other side of the plane, or the plane should be replaced by a rod to which the particle may be attached by a small ring.

The magnitude of R is $\frac{2}{3}W$.

5. A heavy particle, P (Fig. 46), is placed inside a smooth parabolic tube whose axis is vertical, and is acted upon by a horizontal force, F, equal to μPM, PM being the ordinate of the point P; find the position of equilibrium.

Here the forces acting are W, the weight of the particle, R, the normal reaction of the tube, and F. We shall obtain an equation between F and W, without R, by resolving along the tangent at P. If θ = angle between the tangent at P and the vertical,

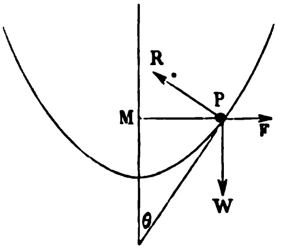

Fig. 46.

$$W\cos\theta = F\sin\theta = \mu y\,.\sin\theta, \text{ where } y = PM.$$

Hence, for the position of equilibrium, retaining the factor $\cos\theta$,

$$\cos\theta\,(W - \mu y\tan\theta) = 0.$$

But if the equation of the parabola is $y^2 = 4mx$, $\tan\theta = \frac{2m}{y}$. Hence the equation is

$$\cos\theta(W - 2\mu m) = 0. \tag{1}$$

This equation of equilibrium can be satisfied in two ways. Firstly, we can have

$$\cos\theta = 0, \tag{2}$$

or $\theta = \frac{\pi}{2}$, which gives the vertex of the tube as the position of equilibrium. This position is *a priori* evident, since the particle would at the vertex be acted upon only by its weight and the reaction of the tube, the force F here being $= 0$.

Secondly, the equation will be satisfied if

$$W - 2\mu m = 0. \tag{3}$$

Now, this is simply a relation between the constants of the problem, and gives no value of θ—that is, no definite position of equilibrium. In fact, if the equation (3) is satisfied, (1) will be satisfied, no matter what θ may be. The result, then, is as follows: if $\mu = \frac{W}{2m}$, the particle will rest in all positions; and if this relation does not hold, the vertex is the only position.

It is well for the student to observe that μ is here the quotient of a force by a line, the force being expressed in the same units as those of W, and the line in the same units as those of PM. For, since we have put $F = \mu PM$, if Q is a force in the same units as those of W, and l a line in the same units as those of PM, it is clear that the proper representation of F would be something of the form

$$Q\frac{PM}{l}; \qquad \therefore \mu = \frac{Q}{l}.$$

6. A heavy particle, resting on a smooth inclined plane, is attached to a string which, passing over a smooth pulley, sustains another heavy particle: find the conditions and position of equilibrium.

Let W be the weight of the particle on the plane, P that of the hanging particle, and θ the inclination of the string to the inclined plane in the position of equilibrium.

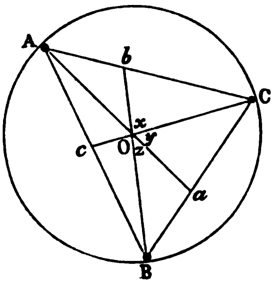

Fig. 47.

For the equilibrium of the particle on the plane, we have, resolving along the plane (since the tension of the string $= P$),

$$W\sin i = P\cos\theta;$$

$$\therefore \cos\theta = \frac{W\sin i}{P}.$$

In order that there may be a position of equilibrium, this value of $\cos\theta$ must be < 1, $\therefore W\sin i$ must be $< P$.

Explain the result when $P = W$.

7. Three particles, whose masses are m_1, m_2, m_3, are placed at three points, A, B, C (Fig. 47), inside a smooth circular tube; they attract or repel

each other with forces directly proportional to their masses and their distances; find the position of equilibrium of the system.

Consider the equilibrium of m_1 at A. It is acted upon by two forces proportional to $m_2 AB$ and $m_3 AC$, in the directions AB and AC. The resultant of these must be normal to the tube at A. But (Cor. 2, p. 21) the resultant acts towards a, the centre of gravity of m_2 and m_3, and if O is the centre, $OB = OC$. Hence $\frac{\sin y}{\sin z} = \frac{m_2}{m_3}$; and, by considering the equilibrium of B, we have $\frac{\sin x}{\sin z} = \frac{m_1}{m_3}$. Therefore $\sin x : \sin y : \sin z = m_1 : m_2 : m_3$. Also $x + y + z = \pi$; therefore x, y, and z are the angles of a triangle whose sides are proportional to m_1, m_2, and m_3. These angles being known from some such equations as $\cos x = \frac{m_2^2 + m_3^2 - m_1^2}{2 m_2 m_3}$, &c., the relative positions of the particles are at once determined. The centre, O, of the tube is the centre of gravity of the particles.

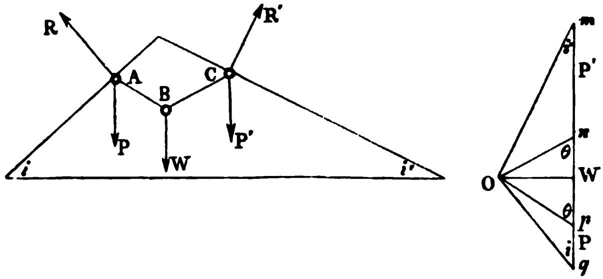

Fig. 48.

8. Two smooth heavy rings, A and C (Fig. 48), slide on two rods which are inclined to the horizon at angles i and i'; a string connecting A and C passes through a smooth heavy ring, B. Find the condition of equilibrium.

Let the weights of A, B, C, be P, W, P', respectively, and let R and R' be the reactions of the rods on A and C. Construct the force-diagram of the system by drawing Om from an arbitrary origin, O, parallel and proportional to R' and mn parallel and proportional to P'; then on will be parallel to BC and proportional to the tension in it. Drawing again np parallel and proportional to W, Op will be parallel to BA, and represent its tension. Finally, if pq represents P, Oq will represent R. Since the tension in ABC is constant, $On = Op$; ∴ a perpendicular from O on mq bisects np. The length of this perpendicular is, on the one hand, $(mn + \frac{1}{2} np) \tan i'$, and on the other, $(pq + \frac{1}{2} np) \tan i$. Hence, equating these, we have

$$(P' + \tfrac{1}{2} W) \tan i' = (P + \tfrac{1}{2} W) \tan i.$$

This is a relation between the constants of the problem, and it therefore constitutes a condition that equilibrium should be at all possible. If this condition is fulfilled, there will be an infinite number of positions of equilibrium. For if θ is the angle which the string BC makes with the vertical, we have from the force-diagram

$$\tan\theta = \frac{W' + 2P'}{W}\tan i';$$

and it can be easily proved that if the two rods are taken as axes of x and y, the locus of B is

$$x \sec(\theta - i') + y \sec(\theta - i) = l \operatorname{cosec}(i + i'),$$

which is a given right line.

9. Two heavy rings, whose weights are P and P' (Fig. 49), rest on the circumference of a smooth vertical circle, and are connected by a weightless string on which a heavy ring, whose weight is Q, slides freely. Find the position of equilibrium.

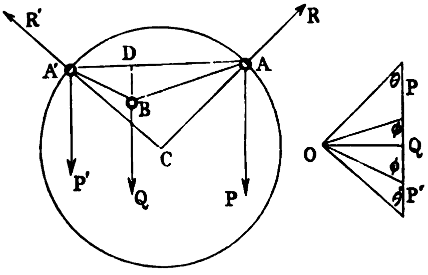

Fig. 49.

Construct the force-diagram. Let θ and θ' be the inclinations of the radii CA and CA' to the vertical, and let ϕ be the inclination of the portions of the string AB and BA' to the vertical.

The force-diagram then gives the statical equations

$$\left(\frac{Q}{2} + P\right)\tan\theta = \left(\frac{Q}{2} + P'\right)\tan\theta'. \qquad (1)$$

$$\left(\frac{Q}{2} + P\right)\tan\theta = \frac{Q}{2}\tan\phi. \qquad (2)$$

To these must be added the geometrical equation which connects the length, l, of the string with the radius, a, of the circle.

Since the horizontal projections of the broken lines ACA' and ABA' are the same, we have

$$a(\sin\theta + \sin\theta') = l \sin\phi. \qquad (3)$$

Equations (1), (2), and (3) are sufficient to determine the unknown angles θ, θ', and ϕ.

10. A body, whose mass is 10 kilogrammes, is supported on a smooth inclined plane by a force of 2 kilogrammes' weight acting along the plane and a horizontal force of 5 kilogrammes' weight; find the inclination of the plane.

Ans. $\sin^{-1}\left(\frac{3}{5}\right)$.

11. A heavy body is sustained on a smooth inclined plane (inclination i) by a force P acting along the plane, and a horizontal force, Q. The inclination being halved, and the forces P and Q each halved, the body is still observed to rest; find the ratio of P to Q.

Ans. $\frac{P}{Q} = 2\cos^2\frac{i}{4}$.

12. A mass of 10 kilogrammes is to be sustained on a smooth inclined plane of 25° inclination by a horizontal force of 5 kilogrammes' weight and a force unknown in magnitude and direction; determine this force in both respects so that there shall be a normal pressure of 2 kilogrammes' weight on the plane.

Ans. The force = 9·07 kilogrammes' weight, and it makes an angle of about 1° 54′ with the normal, having a *downward* component.

13. Find the inclination of a smooth inclined plane if a mass of 24 kilogrammes resting on it is sustained by a horizontal force of 7 and a force of 16 kilogrammes' weight (of unknown direction), while the normal pressure is a force of 15 kilogrammes' weight; find also the unknown direction.

Ans. i = inclination of plane = 53° 53′.
θ = angle made by force with plane = 17° 28′.

14. Find the inclination of a smooth inclined plane if a mass of 20 kilogrammes resting on it is sustained by an up-plane force of 5 and a force of 15 kilogrammes' weight of unknown direction, while the normal pressure is 2 kilogrammes' weight; and find the unknown direction.

Ans. $i = 49° 28′$; $\theta = 47° 9′$.

15. Find the inclination of a smooth inclined plane if a mass of given weight, W, resting on it is sustained by a given horizontal force, P, and a force Q of given magnitude but unknown direction, while the normal pressure is a given force N; find also the unknown direction.

Ans. If, for convenience, $\tan a$ is put for $\frac{P}{W}$, we have

$$\cos(i-a) = \frac{W^2+P^2+N^2-Q^2}{2N\sqrt{W^2+P^2}};$$

$$\sin\theta = \frac{W^2+P^2-N^2-Q^2}{2QN}.$$

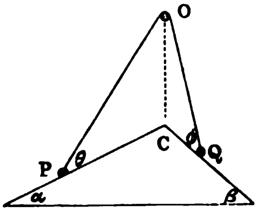

Fig. 50.

16. Two masses, P and Q (Fig. 50), rest on a smooth double-inclined plane, and are attached to the extremities

of a string which passes over a smooth peg, O, at a point vertically over the intersection of the planes, the peg and the masses being in a vertical plane. Find the position of equilibrium.

Ans. If $l =$ the length of the string, and $CO = h$, the position of equilibrium is defined by the equations

$$P\frac{\sin\alpha}{\cos\theta} = Q\frac{\sin\beta}{\cos\phi}, \qquad \frac{\cos\alpha}{\sin\theta} + \frac{\cos\beta}{\sin\phi} = \frac{l}{h},$$

which belong to case (γ), p. 56.

17. Two masses, P and Q, hang from the circumference of a smooth vertical circle by two small rings connected by a string which lies along the convex side of the circle. Find the position of equilibrium, and show that the heavier mass will be higher up on the circle than the lighter.

Ans. If the radius of the circle drawn to P make an angle θ with the vertical diameter, $l =$ length of the string, and $a =$ radius of the circle, the position of equilibrium is defined by the equation

$$P\sin\theta = Q\sin\left(\frac{l}{a} - \theta\right),$$

θ being circular measure.

18. Show, by considering the equilibrium of P and Q (in the last example) as one system, that their centre of gravity lies in the vertical radius of the circle.

19. Two rods are fixed in the same vertical plane at inclinations α and β to the horizon; two rings, whose weights are P and Q, are connected by a string of given length and placed one on each rod; find the position of equilibrium.

Ans. If P is placed on the rod of inclination α, the inclination, θ, of the string to the vertical is given by the equation

$$(P+Q)\cot\theta = P\cot\beta - Q\cot\alpha.$$

20. Two heavy rings, P and Q, connected directly by a string of given length, rest on a smooth circular wire fixed in a vertical plane; find the position of equilibrium.

Ans. If 2α is the angle subtended at the centre of the circle by the string, the inclination, θ, of the string to the vertical is given by the equation

$$(P+Q)\cot\theta = (P-Q)\tan\alpha.$$

21. Two heavy rings, P and Q, connected directly by an elastic string whose tension is proportional to its length[*], rest on a smooth circular wire fixed in a vertical plane; find the position of equilibrium.

Ans. If C is the magnitude of the tension of the string when the string is stretched to the length of the radius of the wire, construct a triangle whose base and two sides are respectively proportional to

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[*] The student will afterwards see that this would be the case if the natural length of the string were so small as to be negligible in the problem.

$\frac{PQ}{C}$, P, Q. Then the base angles of this triangle are those made with the vertical by the radii of the wire drawn to the rings.

22. A mass W is hung from a small ring which can slip over a smooth circular wire fixed in a vertical plane; the ring is also tied to a string which, passing as a chord of the circle over a fixed peg at the top of the circle, sustains a given mass P; find the position of equilibrium and the pressure on the circle.

Ans. If θ is the angle made by the radius drawn to the ring with the vertical, $\sin \frac{1}{2}\theta = \frac{P}{2W}$; and the normal pressure $= W$.

23. In the same vertical plane are fixed a smooth rigid circular wire and a smooth rigid rod; a heavy ring A slips along the circle, and another, B, slips along the rod, these rings being connected by a string of given length; find the position of equilibrium.

Ans. If $r =$ radius of circle; $l =$ length of string AB; $p =$ perpendicular from centre of circle on rod; P and $Q =$ weights of rings A and B, respectively; $\theta =$ angle between radius drawn to A and perpendicular to rod; $\phi =$ angle between AB and perpendicular to rod; $i =$ inclination of rod to horizon, then we have

$$r\cos\theta + l\cos\phi = p,$$
$$Q\cot(\theta - i) + (P+Q)\cot(\phi + i) = P\cot i.$$

The second belongs to the rectilinear case (a) of p. 54, so that θ and ϕ can be constructed from cases (a) and (β).

24. Two very small rings, A and B, capable of slipping along the circumference of a smooth circular wire fixed in a vertical plane, have masses P and Q suspended from them; the rings are attached to the extremities, A, B, of a string ACB which passes over a peg fixed at C vertically over the centre of the circle; find the position of equilibrium.

Ans. Let $l =$ length, ACB, of string; $CA = r$, $CB = r'$; then we have

$$r = \frac{Ql}{P+Q}, \quad r' = \frac{Pl}{P+Q}.$$

25. Two masses rest on the convex side of a parabola whose axis is vertical, and are connected by a string which passes over a smooth peg at the focus; show that equilibrium is impossible unless the weights are equal.

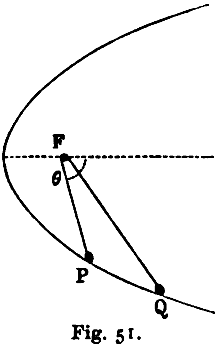

Fig. 51.

26. Two masses, P and Q (Fig. 51), rest on the concave side of a parabola whose axis is horizontal, and are connected by a string which passes over a smooth peg at the focus F. Find the position of equilibrium.

Ans. Let $l =$ length of the string; $\theta =$ the angle which FP makes with the axis; $4m =$ the latus rectum of the parabola; then

$$\cot\frac{\theta}{2} = \frac{P}{\sqrt{P^2+Q^2}}\sqrt{\frac{l-2m}{m}}.$$

27. A particle is placed on the convex side of a smooth ellipse, and is acted upon by two forces, F and F', towards the foci, and a force, F'', towards the centre. Find the position of equilibrium.

Ans. If $r =$ the distance of the particle from the centre of the curve; $b =$ semi-axis minor; and $n = \frac{F-F'}{F''}$; then

$$r = \frac{b}{\sqrt{1-n^2}}.$$

28. A heavy particle, P, is placed on the concave side of a smooth vertical circle whose lowest point is A and highest point B. If the particle is acted upon by two forces, in the directions AP and BP, equal to μBP, and μAP, respectively, find the position of equilibrium.

Ans. Let $W =$ the weight of the particle; $\theta =$ the angle made with the vertical by the radius to P; $a =$ the radius of the circle; then

$$\tan\theta = \frac{2\mu a}{W}.$$

29. A particle, P, is acted upon by two forces towards two fixed points, S and H, these forces being $\frac{\mu}{SP}$ and $\frac{\mu}{HP}$, respectively; prove that P will rest at all points inside a smooth tube in the form of a curve whose equation is $SP.PH = k^2$, k being a constant.

30. A particle, P, is placed inside a smooth circular tube, and acted upon by two forces towards the extremities, A and B, of a fixed diameter, AB; the forces are respectively proportional to PA and PB: prove that the particle will rest in all positions.

31. Two masses, P and Q, connected by a string rest on the convex side of a smooth cycloid. Find the position of equilibrium.

Ans. If $l =$ the length of the string, and $a =$ radius of generating circle, the position of equilibrium is defined by the equation

$$\sin\frac{\theta}{2} = \frac{Q}{P+Q}\cdot\frac{l}{4a},$$

where θ is the angle between the vertical and the radius to the point on the generating circle which corresponds to P.

[The string is supposed to lie along the curve.]

SECTION II.

Rough Curves.

42.] **Friction.** The curves and surfaces which we have hitherto considered were supposed to be incapable of offering resistance in any other than a normal direction. Such curves and surfaces, however, exist only in the abstractions of Rational Statics, and are not to be found in nature. Every surface in nature possesses a certain degree of roughness, in virtue of which it resists the sliding of other surfaces upon it.

Now, there are two ways in which a surface may resist a sliding motion. Firstly, it may possess small inequalities which act as *fixed obstacles* to sliding; and, secondly, there may exist an *adhesion* between its substance and that of another body in contact with it. In virtue of inequalities, the two surfaces get interlocked, and an effort to cause one to slide on the other causes a strain in each of the surfaces, the force which resists this sliding being called *Friction*. Rankine (*Applied Mechanics*, p. 209) distinguishes adhesion from friction on the ground that adhesion between two surfaces is independent of the force by which they are pressed together, and is analogous to *shearing stress*, i.e., to the force (called cohesion) which resists an attempt to divide a solid by causing one part of it to slide on another.

At the same time he holds (*Mechanical Text-Book*, p. 153) that friction is a kind of shearing stress, and this view gives probably the most real and vivid conception of its nature.

43.] **Laws of Friction.** Experiments made by Coulomb and Morin have established the following laws of friction :—

1°. The tangential force necessary to establish the *beginning* of a sliding motion is a constant fraction of the normal pressure between the two surfaces in contact.

2°. With a given normal pressure, the tangential force necessary to establish the beginning of a sliding motion is independent of the extent of the surface of contact.

Subsequent experiments have, however, considerably modified the first of these laws, and shown that it can be regarded only as an approximation to the truth. If N be the normal pressure between the bodies, F the force of friction, and μ the constant ratio of the latter to the former *when slipping is about to ensue*, we have

$$F = \mu N. \qquad (a)$$

The fraction μ in this equation is called *the coefficient of friction*, and if the first law were true, μ would be strictly constant for the same pair of bodies, whatever the magnitude of the normal pressure between them might be. This, however, is not the case. For great differences of normal pressure there are considerable differences in the value of μ. When the normal pressure is nearly equal to that which would crush either of the surfaces in contact, the force of friction increases more rapidly than the normal pressure. Equation (a) is nevertheless very nearly true when the differences of normal pressure are not very great, and in what follows we shall assume this to be the case.

44.] **Causes which Modify the Coefficient of Friction.** Friction being a force called into play by the mutual action of two bodies in contact, μ depends on the particular *pair* of bodies in contact, and is not a quantity pertaining to any *one* body by itself. Moreover, it varies for the same two bodies according as the state of each body varies. Thus, it is not the same for iron and dry oak, as for iron and the same piece of oak with a moistened surface. Neither, again, is it the same for two pieces of wood when their fibres are parallel as when they are perpendicular. In fact, when great accuracy is required, a special experiment should be made to ascertain the coefficient of friction between two bodies which in any case are to act upon and sustain each other. Tables of the coefficient of friction between bodies in specified states are to be found in most practical treatises on Statics.

45.] **Independence of the Extent of the Surface of Contact.** The second law of Friction may at first sight appear strange; but a little reflection will remove objections against its truth. If the total normal pressure between two bodies be N, and the area of the surface of contact S, the pressure per unit of area (which is called the *intensity of pressure*) is $\frac{N}{S}\cdot$ If now, while the normal pressure remains the same as before, the surface of contact is doubled, the pressure per unit of area is only $\frac{N}{2S}$, which is just half as great as before. Hence, though the area over which friction acts is doubled, the intensity of pressure is halved; and it is consistent with common sense that the friction per unit of area should be halved also. Thus, on the whole, the same total tangential force is required to set up sliding in both cases.

46.] **Actual Magnitudes of Coefficients of Friction.** It is well that the student should have some notion of the actual magnitudes of coefficients of friction between bodies. For this purpose he should look at a table of these coefficients. Practically there is no observed coefficient much greater than 1. In Rankine's table the coefficient for damp clay on damp clay is given as 1, and that for shingle on gravel is at the most 1·11. Most of the ordinary coefficients are less than ½.

47.] **Other Coefficients of Friction.** It is found by experiment that the friction which resists the *beginning* of sliding is greater than that which resists its *continuance*. Again, the resistance which is opposed to the rolling of one surface on another is distinguished by the special name of *Rolling Friction*, but it would more properly be called *Resistance to Rolling*. At present we shall limit ourselves to the consideration of the friction of the beginning of motion which is expressed by the equation

$$F = \mu N.$$

48.] **Reaction of a Rough Curve or Surface.** Let AB (Fig. 52) be a rough curve or surface; P the position of a particle on it; and suppose the forces acting on P to be confined to the plane of the paper. Let $N = PR_1 =$ the normal resistance of the surface, acting in the normal, PN, and $F =$ the force of friction, acting along the tangent, PT.

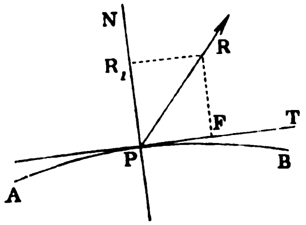

Fig. 52.

The resultant of N and F is a force which we shall call the *Total Resistance* of the surface. It is represented in magnitude and direction by the line $PR = R$, which is the diagonal of the parallelogram determined by N and F. We have seen that the total resistance of a smooth surface is normal; but this limitation does not apply to a rough surface. The angle, ϕ, between R and the normal is given by the equation

$$\tan\phi = \frac{F}{N}.$$

Hence, ϕ will be a maximum when the force of friction bears the greatest ratio to the normal pressure. But this greatest ratio is what we have called the coefficient of friction, μ; and this ratio is attained when the particle is just on the point of slipping along the surface. Therefore *the greatest angle by which the Total Resistance of a rough curve or surface can deviate from the normal is the angle whose tangent is the coefficient of friction for the bodies in contact; and this deviation is attained when slipping is about to commence.*

49.] **Angle of Friction.** The angle between the normal and the total resistance of a rough surface *when slipping is about to take place* is called the *Angle of Friction.* It is sometimes called the *Angle of Repose.* We shall throughout denote it by λ; and if μ is the coefficient of friction,

$$\tan\lambda = \mu.$$

50.] **Experimental determination of μ.** Let* P be the position of a heavy particle, whose weight is W, on a rough plane, AB, whose inclination is gradually increased until P is on the point of slipping down. Consider the equilibrium of P in these circumstances. It is acted upon by *two* forces, namely, its weight, W, and the total resistance, R, of the plane. For equilibrium these forces must be equal and act in opposite senses. Hence R acts in a vertical line; and since slipping is about to take place, the angle between R and the normal, PN, to the plane must (Art. 48) be equal to λ, the angle of friction. But the angle between the vertical and PN is also equal to the inclination of the plane to the horizon. Hence *the inclination of a rough plane on which a particle, acted upon solely by its own weight, is just about to slip, is the Angle of Friction.*

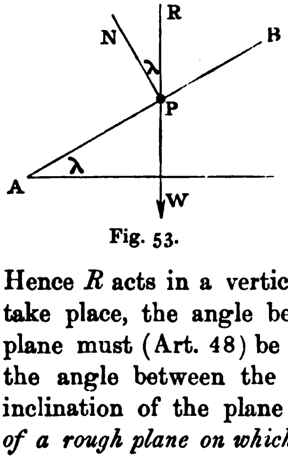

Fig. 53.

This result might have been proved by the resolution of forces. Thus, if N be the normal pressure, the force of friction acting up the plane is μN, since slipping is about to begin. Hence, resolving forces horizontally for the equilibrium of P,

$$N\sin i - \mu N\cos i = 0,$$

i being the inclination; or $\tan i = \mu$; $\therefore i = \lambda$.

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* P ought to be represented in the figure as having a *flat* base in contact with the plane. The student will similarly correct all the subsequent figures.

Morin determined the coefficient of friction between two substances by placing one on a fixed horizontal plane made of the other, and then measuring the least horizontal force which should be applied to the body resting on the plane to cause it to slide. The ratio of this force to the weight of the body is the required coefficient of friction.

51.] **Limitation of the Total Resistance.** As in the case of the resistance of a smooth curve or surface, there is no limit to the *magnitude* of the total resistance of a rough curve or surface —for the surfaces with which we are at present concerned are supposed to be capable of resisting penetration to any extent— the only limitation to which the total resistance is subject being one of *direction*, and this limitation is thus expressed:—

The Total Resistance of a rough curve or surface, though unrestricted in magnitude, can never make with the normal an angle greater than the angle of friction corresponding to the two bodies in contact.

Within this limit, the total resistance can assume any magnitude and direction, so that we at once deduce the following important principle:—

If the Total Resistance can maintain equilibrium, it will do so.

Thus, let P (Fig. 54) be a heavy particle placed upon a rough plane whose inclination is less than λ, the angle of friction. Then it is clear that, to keep P at rest, the total resistance, R, has only to be equal and opposite to W, the weight of P.

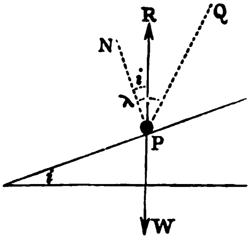

Fig. 54.

But, drawing PQ, making the angle of friction, λ, with the normal, PN, we see that the direction of R falls within the prescribed limit; and therefore the equilibrium will subsist, no matter how great W may be, for there is no limit as to the *magnitude* of R.

52.] **Limiting Equilibrium.** A particle acted upon by any forces and placed upon a rough surface is said to be in *limiting equilibrium* when it is in such a position that the total resistance of the surface makes the angle of friction with the normal. In such a position if any slight change should occur in the circumstances of the particle, in virtue of which the total resistance would be compelled to make a greater angle with the normal, equilibrium could subsist no longer; for the total resistance can never be inclined to the normal at an angle greater than the angle of friction. Or we may put the matter thus. In every case the equilibrium of a particle restricted to a rough curve or surface is broken only by some circumstance which compels the total resistance to make with the normal an angle greater than the angle of friction. The manner in which this is supposed to happen depends on the particular problem. For example, let us enquire into the circumstances of the equilibrium of a heavy particle, whose weight is *W*, on a rough curve, *AB* (Fig. 55), whose plane is vertical, the particle being acted upon by a horizontal force, *F*.

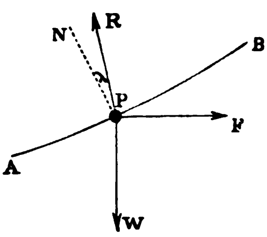

Fig. 55.

The problem proposed for solution may be any one of the three following:—

(*a*) Determine the least horizontal force that will sustain a particle, of weight *W*, at a given point, *P*, of a given rough curve, *AB*.

(*b*) Determine the point at which a particle, of weight *W*, will be just sustained by a given horizontal force, *F*, on a given rough curve, *AB*.

(*c*) Determine the least coefficient of friction that will allow a particle, of weight *W*, to rest at a given point, *P*, of a curve, *AB*, the particle being acted on by a given horizontal force, *F*.

If *PN* be the normal at *P*, and *PR* be drawn making the angle of friction, λ, with it, *PR* will be the direction of the total resistance, since, by supposition, the particle is about to slip down. All three problems are solved by the equation

$$\frac{W}{F} = \cot(\theta - \lambda),$$

θ being the inclination of the tangent at *P* to the horizon. But the manner in which equilibrium is supposed to be broken is not the same in each of them. If, in the first case, $F < W\tan(\theta - \lambda)$, in the second, $\theta > \lambda + \tan^{-1}\left(\frac{F}{W}\right)$, and in the third, $\lambda < \theta - \tan^{-1}\left(\frac{F}{W}\right)$, the particle will not rest at *P*. Thus the equilibrium may be broken by—

(*a*) a slight change in some of the acting forces;

(*b*) a slight change in the position of the particle; or

(*c*) a slight change in the nature of the supporting surface, i. e. a diminution of its roughness.

If the particle is in limiting equilibrium (i. e. if the total resistance makes the angle of friction with the normal to the supporting surface) it is evident that equilibrium will always be broken if the third of these changes occurs; but it may not be broken by either of the others. Take, for example, a heavy particle placed on an inclined plane whose inclination to the horizon is the angle of friction. It is evident that any change may be made, either in its weight or in its position on the plane, and equilibrium will still subsist; for in neither case is the total resistance (equal and opposite to W) compelled to make with the normal an angle $>\lambda$.

In every case of equilibrium it is to be observed that the *Force of Friction* (Art. 43) *acts in the sense opposite to that in which motion would ensue if the bodies in contact became gradually smoother*, and this is the rule for determining the side of the normal at which the Total Resistance acts.

53.] **Friction in non-limiting equilibrium.** The beginner is very prone to assume that, if μ is the coefficient of friction between two bodies, *in every case* in which one of these bodies rests against the other the force of friction is μN, where N is the normal pressure between them. That this is not so he will easily see by considering the case in which a heavy piece of metal rests on a horizontal plane of wood, the coefficient of friction between the metal and the wood being, say, $\frac{2}{5}$, and no forces, other than its weight and the resistance of the plane, acting on the body. So far from the force of friction's being $\frac{2}{5}$ of the normal pressure, the force of friction is zero, and will come into existence only when some horizontal force is applied to the body. The force of friction will always be equal to this horizontal force and will attain the value $\frac{2}{5}N$ only when slipping is about to take place.

The changes both in magnitude and in direction which the Total Resistance between two rough surfaces in contact undergoes while equilibrium changes from a state bordering on motion in one direction to a state bordering on motion in the opposite direction may be very simply illustrated by solving the following problem:—

A heavy body of weight W is held on a rough inclined plane of inclination i by a horizontal force P; the force P being varied gradually from the value required just to sustain the body to the value required just to drag it up the plane, it is required to represent graphically the different magnitudes and directions of the Total Resistance corresponding to the successive values of P.

Let O (Fig. 56) be the position of the body, and measure off a vertical line OW to represent the magnitude of W.

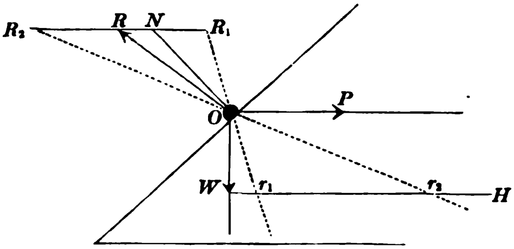

Fig. 56.

Then, for different values of P, the resultant of W and P will be represented by lines drawn from O and terminating on the horizontal line WH. The Total Resistance of the plane on the body is, of course, equal and opposite to the resultant of P and W, and it will therefore be represented by a line drawn from O to a horizontal line, R_1R_2, drawn at the same distance above O as the line WH is below it.

Let ON be the normal to the plane at O, and draw the lines OR_1 and OR_2 making the angle, λ, of friction with the normal at opposite sides of it. Let these lines be produced to meet the line WH in the points r_1 and r_2.

Then, for equilibrium, the resultant of P and W must be represented by some line between Or_1 and Or_2.

When the resultant of P and W is Or_1, the Total Resistance of the plane is OR_1, and since this makes the angle of friction with the normal, the body is on the point of slipping down. When the resultant of P and W is Or_2, the Total Resistance is OR_2, and the body is on the point of slipping up.

The values of P which will just sustain the body and just drag it up are, respectively,

$$W \tan (i-\lambda) \text{ and } W \tan (i+\lambda),$$

as appears at once from the figure or by calculation.

If P has a value between these limits, the Total Resistance, OR, will be intermediate between OR_1 and OR_2, and the equilibrium will not be limiting, i.e. the body will not be on the point of slipping either up or down; and the force of friction, which is the component of R along the plane, will not be μ times the normal pressure, except in the two states bordering on motion.

If P has the value $W \tan i$, which is intermediate between its extreme values, the Total Resistance will be normal to the plane, and in this state there will be *no force of friction* exerted between the plane and the body.

Inclined Plane Diagram. The magnitude of the force which, acting at O in *any* direction, will just sustain or just drag up the body is easily represented in the above figure (Fig. 56). For, draw from the point W a line parallel to the line of action of the force in question, meeting the lines Or_1 and Or_2 in the points f_1 and f_2 (not represented in fig.), respectively; then the lengths Wf_1 and Wf_2 represent the values of the force which will just sustain and just drag up the body, respectively, on the scale which represents the weight by the length OW; and the corresponding Total Resistances are represented by the lines f_1O and f_2O, respectively. In particular, if from W we draw a line parallel to the inclined plane, its intercepts made by Or_1 and Or_2 will give the up-plane forces which will just sustain and just drag up the body. These forces are, respectively,

$$W\frac{\sin(i-\lambda)}{\cos\lambda} \text{ and } W\frac{\sin(i+\lambda)}{\cos\lambda}.$$

54.] **Passive Resistances.** The force of friction between a body and a rough surface belongs to a class of forces called *Passive Resistances*, i.e. forces which come into existence only on account of the action of other forces and which always endeavour to destroy the effect of these other forces. To this class, indeed, belongs also the normal pressure between any two bodies, and also the resistance of air or any other fluid to a body moving through it.

And it is an axiom with regard to all passive resistances that if they can preserve equilibrium they will.

EXAMPLES.

1. A heavy particle is placed on a rough plane inclined to the horizon at an angle less than the angle of friction; find the limits of the direction of the force required to drag it down.

Let PN (Fig. 57) be the normal to the inclined plane, and let PQ be drawn, making the angle $NPQ = \lambda$, the angle of friction. Now, the necessary and sufficient condition that equilibrium should exist is that the resultant of the weight, W, and the force applied, F, should fall within the angle NPQ. Hence, producing NP and QP to n and q, we see that no force applied to P within the angle nPq will disturb the equilibrium. F must, therefore, be applied within the angle NPq, and act from P towards the left of the figure.

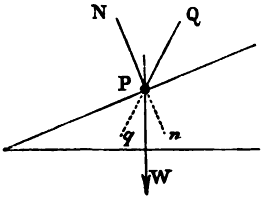

Fig. 57.

2. Two heavy particles, whose weights are P and Q, rest in limiting equilibrium on a rough double-inclined plane, and are connected by a string which passes over a smooth peg at a point, A (Fig. 58), vertically over the intersection, B, of the two planes. Find the position of equilibrium.

Let the inclinations of the planes be α and β; let the length of the string be l, and $AB = h$; and let the portions of the string make angles θ and ϕ with the planes.

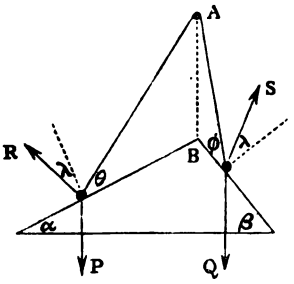

Fig. 58.

Suppose that P is on the point of ascending, and Q of descending. Then, since the motion of each body is about to ensue, the total resistances, R and S, must each make the angle of friction with the corresponding normal; and since the mass P is about to move upwards, R must act towards the left of the normal, while, since Q is about to move downwards, S must act to the left of the corresponding normal.

If T is the tension of the string, we have for the equilibrium of P,

$$T = P\frac{\sin(\alpha+\lambda)}{\cos(\theta-\lambda)}.$$

Again, for the equilibrium of Q,

$$T = Q\cdot\frac{\sin(\beta-\lambda)}{\cos(\phi+\lambda)}.$$

Hence, equating the values of T,

$$P\cdot\frac{\sin(\alpha+\lambda)}{\cos(\theta-\lambda)} = Q\cdot\frac{\sin(\beta-\lambda)}{\cos(\phi+\lambda)}. \qquad (1)$$

This is the only statical equation connecting the given quantities. We obtain a geometrical equation by expressing that AB and the length of the string are given. This is, evidently,

$$h\left(\frac{\cos\alpha}{\sin\theta} + \frac{\cos\beta}{\sin\phi}\right) = l. \qquad (2)$$

Equations (1) and (2) determine the values of θ and ϕ, and constitute the solution of the problem. These equations can be solved graphically, (2) giving the curve in case (γ) of Art. 41, while (1) gives a curve of the fourth degree defined thus—through B (Fig. 41, p. 56) draw an indefinite right line, BE, making the angle $EBA = \lambda$; then P being any point on the curve which represents (1), if AP meets BE in R, we shall have

$$RP = k.PB,$$

where k is a given constant, viz. $\dfrac{P \sin(a+\lambda)}{Q \sin(\beta-\lambda)}$. This locus can be practically constructed with ease thus—Draw any indefinite line, BH, through B; take points M, N, S on this line, in order from B, such that $BM : MN = 1 : k = BS : SN$. Then draw any line ARP through A meeting BE in R; draw NR; from M draw MF, and from S draw SG both parallel to NR and meeting BE in F and G respectively; describe a circle on FG as diameter; then the line AR intersects this circle in points on the required curve.

Other Solution. Instead of considering the total resistances, R and S, we may consider two normal resistances, N and N', and two forces of friction, μN and $\mu N'$, acting respectively down the plane a and up the plane β. In this case, considering the equilibrium of P, and resolving forces along and perpendicular to the plane a, we have

$$\left.\begin{aligned} P\sin a + \mu N &= T\cos\theta, \\ P\cos a &= N + T\sin\theta, \end{aligned}\right\} \qquad \text{(A)}$$

and for the equilibrium of Q,

$$\left.\begin{aligned} Q\sin\beta &= \mu N' + T\cos\phi, \\ Q\cos\beta &= N' + T\sin\phi. \end{aligned}\right\} \qquad \text{(B)}$$

Eliminating N, N', and T from the systems (A) and (B), we arrive at the same statical equation as before.

The method of considering total resistances instead of their normal and tangential components is in many cases more simple than the separate consideration of the latter forces.

3. If in the last question P is given, what are the limits of Q consistent with equilibrium?

If Q be so large that it is about to drag P up, its value, Q_1, will be given by equation (1),

$$Q_1 = P\cdot\frac{\sin(a+\lambda)\cos(\phi+\lambda)}{\sin(\beta-\lambda)\cos(\theta-\lambda)};$$

and if Q be so small that P is about to descend, its value, Q_2, will be

$$Q_2 = P\cdot\frac{\sin(a-\lambda)\cos(\phi-\lambda)}{\sin(\beta+\lambda)\cos(\theta+\lambda)},$$

the angles θ and ϕ being connected by equation (2).

4. A heavy ring is placed on a rough vertical circle; find the limits of its position consistent with equilibrium.

Ans. Draw two diameters making the angle of friction with the vertical diameter. The ring will rest anywhere on the circumference between the two upper extremities, or between the two lower extremities, of these diameters.

5. A body whose mass is 20 kilogrammes is just sustained on a rough inclined plane by a horizontal force of 2 and a force of 10 kilogrammes' weight along the plane; the coefficient of friction is $\frac{2}{5}$; find the inclination of the plane.

Ans. $2\tan^{-1}\left(\frac{25}{48}\right)$.

6. A heavy particle is placed on a rough plane whose inclination to the horizon is $\sin^{-1}\left(\frac{3}{5}\right)$, and is connected by a string passing over a smooth pulley with a particle of equal weight, which hangs freely. Supposing that motion is on the point of ensuing up the plane, find the inclination of the string to the plane, the coefficient of friction being $\frac{1}{2}$.

Ans. By resolving forces along the inclined plane, we have, if $\theta =$ inclination of the string to the plane,

$$\frac{1}{2}\sin\theta = 1 - \cos\theta, \quad \text{or} \quad \frac{1}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin^2\frac{\theta}{2},$$

one solution of which is $\theta = 0$, and the other is $\tan\frac{\theta}{2} = \frac{1}{2}$.

7. In the second solution of the last question, exhibit the position of the string, and explain the result.

8. A heavy particle acted upon by a force equal in magnitude to its weight is just about to ascend a rough inclined plane under the influence of this force; find the inclination of the force to the inclined plane.

Ans. If θ is the required inclination, $\lambda =$ angle of friction, and $i =$ inclination of the plane,

$$\theta = \frac{\pi}{2} - i, \text{ and } \theta = 2\lambda + i - \frac{\pi}{2}$$

are possible solutions. (θ is here supposed to be measured from the *upper* side of the inclined plane. If $\frac{\pi}{2} > 2\lambda + i$, the applied force will act towards the *under* side.)

9. In the first solution of the last question, what is the magnitude of the pressure on the plane?

Ans. Zero. Explain this.

10. What angle must a given force P make with a rough incline so that when a mass of weight W is just sustained, the normal pressure shall be equal to W?

Ans. $\cot^{-1}\frac{\sin i - \mu}{1 - \cos i}$.

11. A mass of 500 kilogrammes can be just sustained on a rough

incline by a horizontal force of 120 kilogrammes' weight, and also (separately) by an up-plane force of 132·6 kilogrammes' weight; find μ and i.

Ans. $\mu = .539$; $i = 41^\circ 51'$.

12. Two masses rest on a rough double inclined plane, being connected by a cord which passes over a smooth pulley at the vertex of the double incline; the inclinations of the planes are 48° and 32°; for both the angle of friction is 28°; if the mass on the first is 560 kilogrammes, and is just on the point of slipping down, calculate the other mass.

Ans. 221 kilogrammes, nearly.

13. A heavy body is to be dragged up a rough inclined plane; find the direction of the least force that will suffice.

Ans. The direction of the force must make the angle of friction with the plane. This follows at once either by resolution of forces or by drawing the *Inclined Plane Diagram*, p. 75; for Fig. 56 shows at once that the least force that will just sustain the body is represented in magnitude and direction by the perpendicular from the point W on the line Or_1, while the least force that will drag the body up is represented by the perpendicular from W on Or_2. In the first case the least force, being at right angles to the Total Resistance, makes the angle of friction with the lower side of the plane, while in the second the least force makes this angle with the upper side.

N.B. This result is often expressed thus:—*the best angle of traction up a rough inclined plane is the angle of friction.*

14. Prove that the horizontal force which will just sustain a heavy particle on a rough inclined plane will sustain the particle on the same plane supposed smooth, if the inclination is diminished by the angle of friction.

15. What is the least coefficient of friction that will allow of a heavy body's being just kept from sliding down an inclined plane of given inclination, the body (whose weight is W) being *sustained* by a given horizontal force, P?

Ans. $\dfrac{W \tan i - P}{W + P \tan i}$.

Explain *à priori*, why we get a negative value for the coefficient of friction unless $W \tan i > P$.

16. It is observed that a body whose weight is known to be W can be just sustained on a rough inclined plane by a horizontal force P, and that it can also be just sustained on the same plane by a force Q up the plane; express the angle of friction in terms of these known forces.

Ans. Angle of friction $= \cos^{-1} \dfrac{PW}{Q\sqrt{P^2 + W^2}}$.

17. It is observed that a force, Q_1, acting up a rough inclined plane will just sustain on it a body of weight W, and that a force, Q_2,

acting up the plane will just drag the same body up; find the angle of friction.

Ans. Angle of friction $= \sin^{-1} \dfrac{Q_2 - Q_1}{2\sqrt{W^2 - Q_1 Q_2}}$.

18. In example 8, p. 61, if the rings A and C are equally rough, find the condition that there may be a limiting equilibrium in which each is about to slip down.

Ans. If λ is the angle of friction, the required condition is

$$\left(P' + \frac{W}{2}\right)\tan(i' - \lambda) = \left(P + \frac{W}{2}\right)\tan(i - \lambda).$$

In this case the lines Om and Oq must be drawn making angles $i' - \lambda$, and $i - \lambda$, respectively, with the line mq.

If the above condition is satisfied, there will be an infinite number of positions of equilibrium, as in ex. 8, p. 61, those of B all lying on a certain right line.

19. In the same example, if one of the rings, C, is in a position of limiting equilibrium, find the direction of the string, the position of the other ring, A, and the direction of the total resistance at it.

Ans. The position of the string is determined by the equation

$$\frac{W}{2} \cdot \tan\theta = \left(\frac{W}{2} + P'\right)\tan(i' \pm \lambda),$$

the + or − sign being used according as C is about to slip up or down. When θ is known, the position of A is known; and the direction of the total resistance at A is found from the equation

$$\left(\frac{W}{2} + P\right)\tan Oqm = \left(\frac{W}{2} + P'\right)\tan(i' \pm \lambda).$$

20. Two small rings, from which hang two masses, P and Q, are fitted on a rough circular wire fixed in a vertical plane, and are connected directly by a string of given length; find the limiting positions of equilibrium.

With the notation of example 20, p. 64, if P is about to descend,

$$(P + Q)\cot\theta = Q\tan(a + \lambda) - P\tan(a - \lambda).$$

21. Two masses, P and Q, hang from two small rings, A and B, fitted on a rough circular wire fixed in a vertical plane, the rings being connected by a string passing along the circumference; find the limits of the position of equilibrium, supposing no friction between the string and the wire.

Ans. If θ be the angle made by the radius to A with the vertical, $l =$ the length of the string, and $a =$ the radius of the circle, θ may have any value between θ_1 and θ_2, these being given by the equations

$$\tan\theta_1 = \frac{Q\sin\left(\frac{l}{a} + \lambda\right) + P\sin\lambda}{Q\cos\left(\frac{l}{a} + \lambda\right) + P\cos\lambda},$$

$$\tan\theta_2 = \frac{Q\sin\left(\frac{l}{a}-\lambda\right)-P\sin\lambda}{Q\cos\left(\frac{l}{a}-\lambda\right)+P\cos\lambda},$$

λ being the angle of friction.

22. If the wire in the last question is in the form of any curve, show that in the *limiting* positions of equilibrium the total resistances at A and B intersect on the circle passing through A, B, and the point of intersection of the normals at A and B.

23. Two heavy particles, P and Q (Fig. 59), rest, one on a rough diameter, AB, of a rough vertical circle, and the other on the convex side of the circle, the particles being connected by a string which passes over a smooth peg at the upper extremity, B, of the diameter. Find the position of equilibrium, the string being supposed to be nowhere in contact with any rough surface, and the coefficients of friction for P and Q being different.

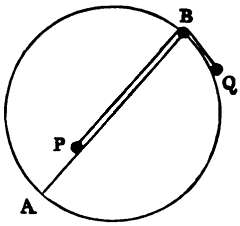

Fig. 59.

Ans. If α = the inclination of AB to the vertical, θ = inclination of the radius drawn to Q to the vertical, μ = coefficient of friction between P and AB, μ' = coefficient of friction between Q and the circle, the limiting positions of equilibrium are given by the equations

$$Q(\sin\theta_1+\mu'\cos\theta_1) = P(\cos\alpha-\mu\sin\alpha),$$
$$Q(\sin\theta_2-\mu'\cos\theta_2) = P(\cos\alpha+\mu\sin\alpha).$$

24. Three particles of weights w_1, w_2, w_3 rest on a rough horizontal table; w_1 is connected with w_2 by a string fully stretched, and w_2 is similarly connected with w_3. Find the magnitude and direction of the least force which, applied to w_3, will move all the particles.

Ans. For the possibility of the motion the angle between the line $w_2\,w_3$ and the line $w_1\,w_2$ produced through w_2 must be acute. Take any point, O, and through it draw OA parallel to the line $w_2\,w_1$ and proportional to μw_1; from A draw AB parallel to the line $w_2\,w_3$, and inflect OB proportional to $\mu\,w_2$; draw OB' equal and parallel to AB, and draw OC perpendicular to OB' towards AB and proportional to μw_3. Then the diagonal through O of the rectangle determined by OC and OB' gives the required direction and magnitude of the force to be applied to w_3.

25. If in the last example, w_1 and w_2, instead of being connected with each other, are each connected with w_3, find the direction and magnitude of the least force which, applied to w_3, will move them all.

26. Any number of particles of weights w_1, w_2, w_3, ... w_n, lie on a rough horizontal table, w_1 and w_2 being connected by a tight string, as also w_2 and w_3, w_3 and w_4, and so on. Find the magnitude and direction of the least force which, applied to the last particle, w_n, will

cause the whole set to move simultaneously, and find the conditions that such movement shall be possible.

Ans. Take any point O, and draw OA_{12} parallel to the string $w_1 w_2$ and proportional to μw_1; draw OA_{23} parallel to the string $w_2 w_3$, and inflect $A_{12}A_{23}$ proportional to μw_2; draw OA_{34} parallel to the string $w_3 w_4$, and inflect $A_{23}A_{34}$ proportional to μw_3; and so on, until the vertex $A_{n-1,n}$ is reached; then draw $A_{n-1,n}P$ perpendicular to $OA_{n-1,n}$ and proportional to μw_n. The line OP represents the required force in magnitude and direction. The lines OA_{12}, OA_{23}, OA_{34}, ... represent the tensions of the strings; and for the possibility of the motion the angles $OA_{12}A_{23}$, $OA_{23}A_{34}$, $OA_{34}A_{45}$ ... must each be $> \frac{\pi}{2}$.

27. Two heavy particles, P and Q (Fig. 60), rest on two rough circular arcs which have a common vertical tangent at O; P and Q are connected by a string which passes over a smooth pulley at O; find the positions of limiting equilibrium.

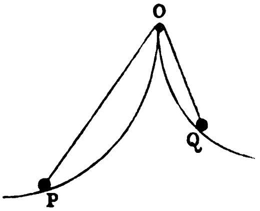

Fig. 60.

Ans. Let θ and ϕ be the angles subtended by the arcs OP and OQ at the centres of the corresponding circles, a and b the radii of the circles, λ and ϵ the angles of friction for P and Q, respectively, and l the length of the string; then, if P is about to slip down, the equations

$$P\frac{\cos(\theta+\lambda)}{\cos\left(\frac{\theta}{2}+\lambda\right)} = Q\frac{\cos(\phi-\epsilon)}{\cos\left(\frac{\phi}{2}-\epsilon\right)},$$

and

$$a\sin\frac{\theta}{2}+b\sin\frac{\phi}{2}=\frac{l}{2},$$

determine the position of equilibrium. Changing the signs of λ and ϵ, we obtain the position in which Q is about to slip down.

[Instead of particles on the circular arcs, we may suppose small rings from which the masses P and Q are suspended.]

28. A particle rests on a rough curve whose equation is $f(x, y) = 0$, and is acted on by forces the sums of whose components along the axes of x and y are X and Y; prove that the particle will rest at all points on the curve at which

$$\frac{X\frac{df}{dx}+Y\frac{df}{dy}}{\sqrt{X^2+Y^2}\,.\,\sqrt{\left(\frac{df}{dx}\right)^2+\left(\frac{df}{dy}\right)^2}} > \cos\lambda.$$

29. Two rings whose weights are P and Q are moveable on a rough rod inclined to the horizon at an angle i; these rings are connected by a string of given length which passes through and supports a smooth heavy ring W; find the greatest distance between P and Q.

Ans. If θ is the inclination of either portion of the string to the vertical, the greatest distance between the rings is obtained by giving $\tan\theta$ the less of the values

$$\frac{W+2Q}{W}\tan(\lambda-i),\quad \frac{W+2P}{W}\tan(\lambda+i),$$

Q being the upper ring.

30. DE and EF are two rough inclined planes intersecting at right angles at E (forming a double inclined plane); on these planes are placed two given masses, P and Q, respectively, connected by a cord over a pulley at E, the coefficients of friction being the same for both. P is on the point of dragging Q up; but if the plane EF is removed, and Q allowed to hang vertically, Q is on the point of dragging P up. Prove the following construction for finding the values of the inclination, i, of DE and the angle, λ, of friction: draw a right line OA proportional to Q, and on OA as diameter describe a circle, whose centre is B; at B draw a line BC proportional to P and making the angle $OBC = \tan^{-1}\frac{Q}{P}$; on BC as diameter describe a circle; then find a point R on the first circle such that if BR meets the second circle in S, we shall have $OR = CS$. The angle $AOR = \lambda$, and $i = \lambda + OBC$.

[If $P = 150$ and $Q = 140$, scale measurement and subsequent correction will give $\lambda = 11^\circ 31'$, and $i = 54^\circ 32'$.]

31. A mass W is placed on a face of a rough wedge inclined at an angle i to the ground, on which the wedge rests, the mass of the wedge being W', the coefficient of friction between W and the wedge being μ, and that between the wedge and the ground being μ'. If a horizontal rope is attached to W and pulled with gradually increasing force, find how motion will ensue.

[Thus, if $W = 200$, $W' = 96$, $\mu = \frac{1}{4}$, $\mu' = \frac{1}{2}$, $\tan i = \frac{5}{12}$, the two bodies will move as one.]

32. A mass on a rough inclined plane is attached by a tight cord to a fixed point in the plane, the direction of the cord making a given angle, θ, with the line in which the plane intersects the ground; if the inclination of the plane is gradually increased, when will the body begin to slip?

Ans. When $\tan i = \frac{\mu}{\cos\theta}$, where μ is the coefficient of friction.

CHAPTER IV.

THE PRINCIPLE OF VIRTUAL WORK.

SECTION I.

A Single Particle.

55.] **Orthogonal Projection.** Let Ox and AB (Fig. 61) be any two right lines inclined at an angle θ. If from the extremities, A and B, of the right line AB, two perpendiculars, Aa and Bb, be let fall on Ox, the line ab is called the *orthogonal projection* of AB on Ox. If the lines Aa and Bb had been each drawn parallel to a given line, which is not perpendicular to Ox, ab would be an *oblique* projection of AB.

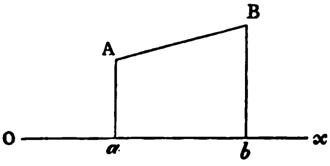

Fig. 61.

In the case of orthogonal projection it is evident that $ab = AB\cos\theta$.

56.] **Projection of a Broken Line.** Let $ABCD$ (Fig. 62) be a zigzag or broken line. Then it is evident that the projection (orthogonal or oblique) of the line AD, joining the first and last points, A and D, is equal to the sum of the projections of the separate lines, AB, BC, and CD, on any line Ox.

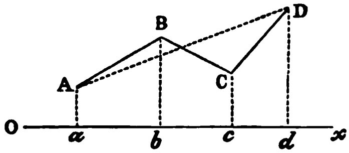

Fig. 62.

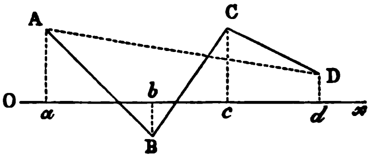

Fig. 63.

This is also true when the line Ox, on which the projection takes place, cuts any or all of the lines AB, BC, ... between the vertices, A, B, C, ..., of the polygon formed by them, as in Fig. 63.

If the sides of a closed polygon taken in order be marked with arrows pointing from each vertex to the next one, and if their projections be marked with arrows flying in the same directions, then, lines measured from left to right being considered positive, and lines from right to left negative, we may evidently state this result as follows :—

The sum of the projections of the sides of a closed polygon on any right line, allowance being made for positive and negative projections, is zero.

57.] **Virtual Displacement. Virtual Work.** If a point at O (Fig. 64) be conceived as displaced to A, OA may be called the *virtual displacement* of the point.

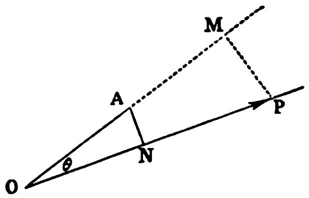

Fig. 64.

Let OI be the direction of a force, P, and let AN be drawn perpendicular to it; then ON is the projection of the virtual displacement along OP, and the product of the force, P, by the projection, ON, of the virtual displacement is called the *virtual work* of the force. We shall therefore say that—

The VIRTUAL WORK *of a force is the product of the force and the projection along its direction of the Virtual Displacement of its point of application.*

If θ be the angle between the force and the virtual displacement,

$$\text{The Virtual Work} = P.ON = P.OA\cos\theta = P\cos\theta.OA.$$

Now $P\cos\theta$ is the projection of the force along the direction of displacement, and is equal to OM, if PM is perpendicular to OA. Hence we may also define the virtual work of a force as follows :—

The virtual work of a force is the product of the virtual displacement of its point of application and the projection (or component) of the force in the direction of this displacement.

This latter definition is for some purposes more convenient than the former. It is to be observed that the projection of a line AB (Fig. 61), of given length, remains unaltered in magnitude when AB is moved parallel to itself into any position.

58.] **Theorem.** *The virtual work of a force is equal to the sum of the virtual works of its components, rectangular or oblique.*

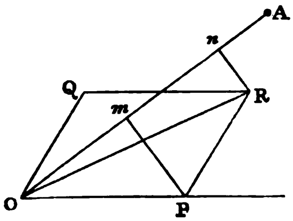

Fig. 65.

Let a force R, represented by OR (Fig. 65), act at O, and let its components be P and Q, represented by OP and OQ. Let OA be the virtual displacement of O, and let its projections on R, P, and Q, be r, p, and q, respectively. Then the virtual works of these forces are $R.r$, $P.p$, $Q.q$. Draw Pm and Rn perpendicular to OA. Then On is the projection of R in the direction of the displacement, and by the end of Art. 57,

$$R.r = OA \times On.$$

Similarly $P.p = OA \times Om$, and $Q.q = OA \times mn$.

Hence

$$P.p + Q.q = OA(Om + mn) = OA \times On = R.r.\text{—Q. E. D.}$$

59.] **Theorem.** *The sum of the virtual works of any number of forces acting at a point is equal to the virtual work of the resultant.*

This may be proved by taking the forces two-and-two, and using the last Theorem, or by making use of the polygon of forces (see Fig 11, p. 23). The sum of the virtual works of the forces is equal to the virtual displacement multiplied by the sum of the projections along it of the sides of the polygon parallel to the forces (Art. 57). But (Art. 56) the sum of these projections is equal to the projection of the remaining side of the polygon, and this side represents the resultant. Therefore, &c.

It follows, then, that—

When a system of forces acting at a point is in equilibrium, the sum of the virtual works of the forces $= 0$.

For, such a system will be represented by a closed polygon, and (Art. 56) the sum of the projections of the sides of the polygon along any right line $= 0$.

60.] **Convention of Signs.** If the virtual displacement, OA (Fig. 66), project on the line of the force P in a sense *opposite* to that in which P acts, the projection ON is to be considered negative, and the virtual work is negative. In this case P will also project on the line of displacement in a sense opposite to OA.

In Fig. 67 the virtual displacement, *OA*, is such as to give positive projections, *Or* and *Op*, along the forces *R* and *P*, and a negative projection, *Oq*, along *Q*. And if in this case the lengths of *Or*, *Op*, and *Oq* are denoted by *r*, *p*, and *q*, the equation of virtual work will be $R \cdot r = P \cdot p - Q \cdot q$.

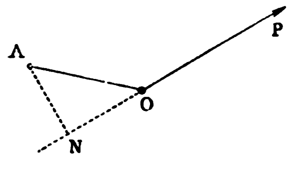

Fig. 66.

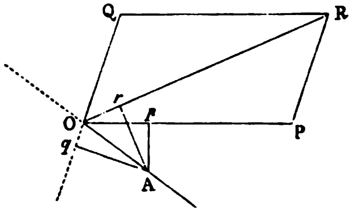

Fig. 67.

61.] **Nature of the Displacement.** It must be carefully observed that the displacement of the particle on which the forces act is both VIRTUAL and perfectly ARBITRARY. In the motion of the particle, treated of in Kinetics, the displacement is often taken to be that which the particle *actually* undergoes; but in the statical problem of the equilibrium of forces, the relation between them, expressed in an equation of virtual work, holds, whatever the displacement may be—that is, it holds whether the displacement be an actual or merely an imagined one. The second characteristic of the displacement, namely its *arbitrariness*, is most important, as will presently appear.

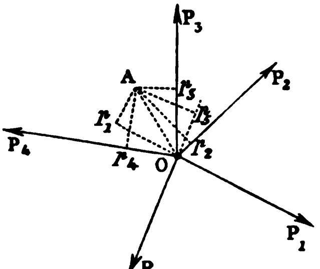

Fig. 68.

62.] **General Equation of Virtual Work.** Let several forces, P_1, P_2, ...(Fig. 68), act in equilibrium on a particle, *O*, and let *OA* be any conceived, or *virtual*, displacement of *O*. Letting fall perpendiculars, Ap_1, Ap_2, ..., on the forces, the projections Op_2, Op_3, and Op_4, are all positive, while Op_1 and Op_5 are negative (Art. 60). Hence the equation of virtual work is

$$-P_1 \cdot Op_1 + P_2 \cdot Op_2 + P_3 \cdot Op_3 + P_4 \cdot Op_4 - P_5 \cdot Op_5 = 0.$$

If the projections of the displacement be denoted by p_1, p_2, ..., and if these quantities are supposed to carry their proper signs with them, this equation becomes, the number of forces being any whatever,

$$P_1 . p_1 + P_2 . p_2 + P_3 . p_3 + \ldots = 0, \tag{1}$$

or

$$\Sigma (P . p) = 0. \tag{2}$$

63.] **General Displacement of a Particle.** The most general displacement of a single particle is a simple motion of translation from the point, O, which it occupies, to another point, A. It is true that in Molecular Dynamics, very small portions of matter are conceived as capable not only of translations but also of rotations about axes through themselves. Indeed every portion of matter, since it must possess extension in space, must be capable of both kinds of displacement; but any consequences resulting from rotational displacement of a particle may be at present entirely neglected.

64.] **Deduction of the Equations of Equilibrium from the Equation of Virtual Work.** Through O draw any two axes, Ox and Oy, rectangular or oblique, and let a and β be the projections of the virtual displacement, OA, along these axes. Replace the force P_1 by its components, X_1 and Y_1, along Ox and Oy. Then (Art. 58)

$$P_1 . p_1 = aX_1 + \beta Y_1.$$

Similarly,

$$P_2 . p_2 = aX_2 + \beta Y_2,$$

$$P_3 . p_3 = aX_3 + \beta Y_3.$$

. . . . . .

. . . . . .

Hence equation (1) of Art. 62 becomes

$$a(X_1 + X_2 + X_3 + \ldots) + \beta (Y_1 + Y_2 + Y_3 + \ldots) = 0,$$

or

$$a\Sigma X + \beta \Sigma Y = 0. \tag{1}$$

Now a and β are perfectly independent of each other. For, the displacement OA may be chosen so as to keep a constant while varying β at pleasure, or *vice versâ*. Suppose, then, that β' and a are the projections of a new virtual displacement, and we shall have

$$a\Sigma X + \beta' \Sigma Y = 0. \tag{2}$$

Subtracting (2) from (1), we have

$$(\beta - \beta') \Sigma Y = 0.$$

Now $\beta - \beta'$ is not $= 0$, therefore ΣY must $= 0$; and in the same way $\Sigma X = 0$. Hence we arrive at the equations of resolution of forces

$$\Sigma X = 0, \ \Sigma Y = 0,$$

which were deduced in Chap. II.*

65.] **Elementary Virtual Work.** In the general equation of virtual work, for forces acting in equilibrium on a *single* particle, namely,

$$P_1 . p_1 + P_2 . p_2 + P_3 . p_3 + \ldots = 0, \text{ or } \Sigma (P . p) = 0,$$

no limitation has been placed upon the magnitude of the virtual displacement. This equation is true, independently of its magnitude; but it is generally more convenient to assume the virtual displacement to be infinitesimal, even in the case of the equilibrium of a single particle, and it is absolutely necessary to do so (as will presently be seen) in treating of the equilibrium of a connected system of particles.

If the virtual displacement is infinitesimal, its projections, $p_1, p_2, \ldots$, on the several forces acting upon the particle are all infinitesimal. We shall, therefore, denote these small projections in future by $\delta p_1, \delta p_2, \ldots$, and the equation of elementary virtual work will be

$$P_1 . \delta p_1 + P_2 . \delta p_2 + P_3 . \delta p_3 + \ldots = 0,$$

or

$$\Sigma P \delta p = 0.$$

66.] **Case in which the Virtual Work of a Force vanishes.** If a force P act at a point O, and if the virtual displacement OA is at right angles to the direction of P, it is clear that δp, the projection of OA on the direction of P, is equal to zero. Hence, *when the virtual displacement is at right angles to the direction of the force, the virtual work of the force* = 0, *and the force will not enter into the equation of virtual work.* Such a virtual displacement is always a convenient one to choose when we desire to get rid of some unknown force which acts upon a particle or a system. For example, let a particle, O, of weight W, be sustained on a smooth inclined plane by a force, P, making an angle

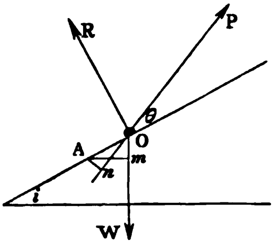

Fig. 69.

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* These equations are, of course, implied in the proof of the principle of virtual work (Art. 59).

θ with the plane. If we wish to find the magnitude of P in terms of W, without bringing the unknown reaction, R, into our equation, we conceive O as receiving a virtual displacement, OA (the magnitude of which is, in the present case, unlimited), at right angles to R, that is, along the plane. Drawing Am and An perpendicular to W and P, respectively, the equation of virtual work is

$$W\,.\,Om - P\,.\,On = 0.$$

But $Om = OA\,.\sin i$, and $On = OA\,.\cos\theta$; therefore

$$W\sin i - P\cos\theta = 0.$$

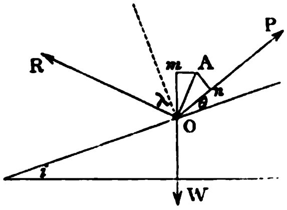

Fig. 70.

As a second example, let us suppose that the plane is rough, and that the particle is on the point of being dragged up the plane. The normal resistance will then be replaced by the total resistance, R, inclined to the normal at an angle $= \lambda$, the angle of friction. Let the virtual displacement, OA (Fig. 70), now take place perpendicularly to R. Then the equation of virtual work is

$$-W\,.\,Om + P\,.\,On = 0.$$

But $Om = OA\,.\sin(i+\lambda)$, and $On = OA\,.\cos(\lambda-\theta)$; therefore

$$W.\sin(i+\lambda) = P\cos(\lambda-\theta).$$

As a third example, let us find the horizontal force which is necessary to keep a heavy particle in a given position inside a smooth circular tube (Fig. 71).

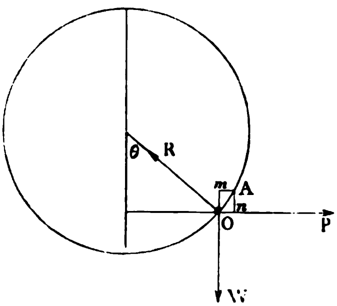

Fig. 71.

Let the virtual displacement, OA, be an indefinitely small one $= ds$, along the tube. Then since ds is infinitesimal, the projection of OA on R will be zero. Also $Om = ds\,.\sin\theta$, and $On = ds\,.\cos\theta$; therefore the equation of virtual work is

$$-Wds\,.\sin\theta + Pds\,.\cos\theta = 0,$$

or $$P = W\tan\theta.$$

If the tube is rough, and the particle in limiting equilibrium, instead of the normal reaction we must draw the total resistance, making the angle λ with the normal at the right or left hand side, according as P is the force which *just sustains* the particle, or the force which will *just drag* it up the tube, and take the virtual displacement, not along the tube, but at right angles to the total resistance. In this case we obtain

$$P = W \tan(\theta \mp \lambda).$$

67.] **Condition of Equilibrium of a Particle as determined by the Principle of Virtual Work.** It will now be sufficiently clear that—

For the equilibrium of a free particle acted on by any forces in one plane it is necessary and sufficient that the virtual work of the system of forces for every arbitrary displacement whatsoever should vanish.

First, it is *necessary* that the virtual work should vanish for every displacement. For, the sum of the virtual works of the forces is equal to the virtual work of their resultant, and if this sum did not vanish, the resultant force could not vanish, and therefore the particle could not be in equilibrium.

Secondly, it is *sufficient* that this sum should vanish for every displacement. This sum is equal to the virtual work of the resultant, and if this vanishes for all possible displacements, the resultant force itself must be zero, and therefore the particle is at rest. For, if possible, let there be a resultant R, which is not zero. Then, since the virtual displacement is quite arbitrary, we may choose it so that it gives a projection $= \delta r$ (which is not $= 0$) on the direction of R. Now, since the virtual work of the system vanishes, we have $R\delta r = 0$.
But since δr is not $= 0$, R must be $= 0$, and the particle is, therefore, at rest.

68.] **Normals to Curves.** The equation of virtual work furnishes a ready method of drawing normals to certain curves. For example, to draw a normal at any point, O, of an ellipse (Fig. 72) let a particle be placed at O inside a smooth elliptic tube whose foci are F and F', and let it be kept in equilibrium by two forces, P and P', directed towards the foci. Let $OF = r$, $OF' = r'$. Then by the property of the ellipse,

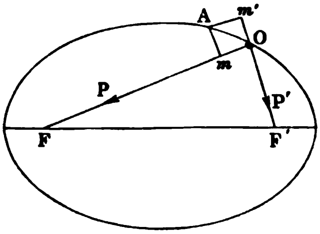

Fig. 72.

$$r + r' = \text{a constant}.$$

Hence, proceeding to a close point, A, we have

$$\delta r + \delta r' = 0. \tag{1}$$

Now the resultant of P and P' is normal to the curve, and is destroyed by the normal reaction. Drawing Am and Am' perpendicular to P and P', the equation of virtual work is

$$P \,.\, Om - P' \,.\, Om' = 0.$$

But $Om = -\delta r$, and $Om' = \delta r'$; therefore this equation becomes

$$P . \delta r + P' . \delta r' = 0. \tag{2}$$

Equation (1) gives $\delta r' = -\delta r$; therefore, substituting in (2), we have

$$P = P',$$

or the forces towards the foci must be equal. But the resultant of two equal forces bisects the angle between them.

Hence the normal at any point of an ellipse bisects the angle between the focal radii drawn to the point.

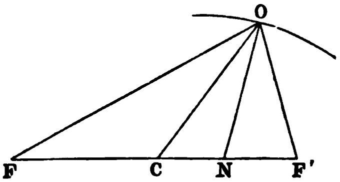

Fig. 73.

Again, the ovals of Cassini are given by the equation

$$rr' = k^2,$$

r and r' being the distances of a point, O, on the curve (Fig. 73), from two fixed points, F and F'. If two forces, P and P', act at O towards F and F', their resultant being normal to the curve, we have for a small virtual displacement along the curve

$$P\delta r + P'\delta r' = 0. \tag{1}$$

But, differentiating the equation of the curve,

$$r'\delta r + r\delta r' = 0. \tag{2}$$

Hence from (1) and (2)

$$\frac{P}{P'} = \frac{r'}{r}.$$

Now, if C is the middle point of FF', we have

$$\frac{r'}{r} = \frac{\sin F}{\sin F'} = \frac{\sin COF}{\sin COF'}.$$

Therefore
$$\frac{P}{P'} = \frac{\sin COF}{\sin COF'}.$$

But if ON be the direction of the resultant,

$$\frac{P}{P'} = \frac{\sin NOF'}{\sin NOF}.$$

Hence $NOF' = COF$; and the normal is, therefore, constructed by joining the point O, on the curve, to the middle point of the line joining the foci, F and F', and then drawing the right line ON so that $\angle NOF' = \angle COF$. The line ON is the normal at O.

EXAMPLES.

1. If the equation of a curve is expressed in the form $\frac{r}{r'} = k$, k being a constant, and r, r' the distances of any point on the curve from two fixed points, A, B, show that the normal to the curve divides AB externally in the ratio $k^2 : 1$, and that the curve is therefore a circle.

2. Prove that the normal to the curve $\frac{1}{r^n} + \frac{1}{r'^n} = \frac{1}{a^n}$ divides AB in the ratio $\left(\frac{r}{r'}\right)^{n+2}$.

3. Give a simple construction for the normal to a Cartesian oval, whose equation is $lr + mr' = a$.

4. The equation of the magnetic curve is $\cos\omega + \cos\omega' = k$ (example 37, p. 51). If N and S are the poles, prove that the normal at a point P is constructed by measuring, on lines perpendicular to PN and PS, lengths proportional to PS^2 and PN^2, respectively, and proceeding as in last Article.

5. The equation of any curve being $f(r, r') = 0$, prove that if the normal is constructed by measuring constant lengths, Pa and Pb, from a point P on the curve, along the lines PA and PB, the curve must belong to the Cartesian ovals.

[This follows at once from the integral of the equation $\frac{df}{dr} = k\frac{df}{dr'}$; for this integral gives $f = \phi(kr + r')$; therefore all such curves give $kr + r' = \text{const.}$]

6. Show that for curves given by the equation $f(\omega, \omega') = 0$, a construction similar to that in the last example (except that the constant lengths are measured on perpendiculars to PA and PB) will hold only when the equation is

$$\tan^n\frac{\omega}{2}\tan^m\frac{\omega'}{2} = k.$$

[This follows from the integral of the equation

$$\frac{1}{r}\frac{df}{d\omega} = \frac{k}{r'}\frac{df}{d\omega'}, \quad \text{or} \quad \sin\omega\frac{df}{d\omega} = k\sin\omega'\frac{df}{d\omega'},$$

for the method of obtaining which integral see Boole's Differential Equations, p. 328].

7. Apply the result in the last example to construct the normal to an ellipse at any point.

[The equation of the ellipse is $\tan\frac{\omega}{2}\cdot\tan\frac{\omega'}{2}=k$.]

The general theorem* of which these are particular cases is the following:—Let the equation of any curve be expressed in the form

$$f(r_1, r_2, r_3, \ldots r_n) = 0,$$

where $r_1, r_2, r_3, \ldots r_n$, denote the distances of any point, P, (Fig. 74), on the curve, from a number of fixed points, $A_1, A_2, A_3, \ldots A_n$; then, if on $PA_1, PA_2, PA_3, \ldots PA_n$, we measure off lengths $Pa_1, Pa_2, Pa_3, \ldots Pa_n$ proportional to

$$\frac{df}{dr_1}, \frac{df}{dr_2}, \frac{df}{dr_3}, \ldots \frac{df}{dr_n},$$

and find G, the centroid ('centre of gravity') of the points $a_1, a_2, a_3, \ldots a_n$, PG will be the normal to the curve at P. [f is used for shortness instead of $f(r_1, r_2, r_3, \ldots r_n)$].

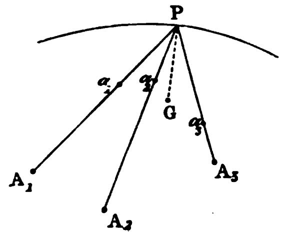

Fig. 74.

The proof of this theorem is exceedingly simple from a statical point of view. Suppose a number of forces, $P_1, P_2, P_3, \ldots P_n$, to act at P along the lines $PA_1, PA_2, PA_3, \ldots PA_n$; then these forces will have a resultant normal to the curve if

$$P_1\delta r_1 + P_2\delta r_2 + P_3\delta r_3 \ldots + P_n\delta r_n = 0.$$

But $$\frac{df}{dr_1}\delta r_1 + \frac{df}{dr_2}\delta r_2 + \frac{df}{dr_3}\delta r_3 \ldots + \frac{df}{dr_n}\delta r_n = 0;$$

hence if $$P_1 : P_2 : P_3 : \ldots P_n = \frac{df}{dr_1} : \frac{df}{dr_2} : \frac{df}{dr_3} : \ldots \frac{df}{dr_n},$$

the resultant acts in the direction of the normal. The rest easily follows by Leibnitz's graphic method of representing the resultant of any number of concurrent forces (see p. 21).

This theorem may be extended to curves given by equations of the form

$$f(\omega_1, \omega_2, \omega_3, \ldots \omega_n) = 0,$$

---

* This theorem is, I believe, due to Tschirnhausen.

where $\omega_1, \omega_2, \omega_3, \ldots \omega_n$ are the angles which PA_1, PA_2, PA_3, $\ldots PA_n$ make with a fixed line.

Let forces $Q_1, Q_2, Q_3, \ldots Q_n$, act at P perpendicularly to the lines $PA_1, PA_2, PA_3, \ldots PA_n$. Then the virtual work of Q_1 for a displacement along the curve is evidently $Q_1 r_1 \delta\omega_1$. Hence the forces will have a resultant normal to the curve if

$$Q_1 r_1 \delta\omega_1 + Q_2 r_2 \delta\omega_2 + Q_3 r_3 \delta\omega_3 \ldots + Q_n r_n \delta\omega_n = 0.$$

But $$\frac{df}{d\omega_1}\delta\omega_1 + \frac{df}{d\omega_2}\delta\omega_2 + \frac{df}{d\omega_3}\delta\omega_3 \ldots + \frac{df}{d\omega_n}\delta\omega_n = 0;$$

therefore the resultant will be normal if

$$Q_1 : Q_2 : Q_3 : \ldots Q_n = \frac{1}{r_1}\frac{df}{d\omega_1} : \frac{1}{r_2}\frac{df}{d\omega_2} : \frac{1}{r_3}\frac{df}{d\omega_3} : \ldots \frac{1}{r_n}\frac{df}{d\omega_n}.$$

Consequently, the rule is—measure off lengths, Pb_1, Pb_2, &c., proportional to $\frac{1}{r_1}\frac{df}{d\omega_1}$, $\frac{1}{r_2}\frac{df}{d\omega_2}$, &c., on lines drawn at P perpendicularly to PA_1, PA_2, &c., in the directions in which the angles ω_1, ω_2, &c., increase; find the centroid of the points, b_1, b_2, &c.; then the line joining this point to P is the normal to the curve.

Section II.

A System of two Particles.

69.] Projection of a Displaced Line of Constant Length. Let a line, AB (Fig. 75), be a right line which is displaced into any close position, $A'B'$, its length remaining constant. Let $\delta\theta$ be the small angle between AB and $A'B'$, and let ab be the projection of $A'B'$ on its original position. Then Aa, the projection of the displacement AA', is equal to Bb, the projection of the displacement BB', if infinitesimals of a higher order than the first are neglected.

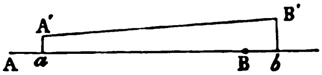

Fig. 75.

For, $$ab = A'B'.\cos(\delta\theta) = A'B'\left(1 - \frac{(\delta\theta)^2}{1.2} + \ldots\right).$$

Hence the difference between ab and $A'B'$ (or AB) is of the order of $(\delta\theta)^2$; and therefore, rejecting $(\delta\theta)^2$, we have

$$AB = ab,$$
$$\therefore\ Aa = Bb.$$

The result may be thus stated:—the difference between AB and ab is infinitesimal compared with the greatest displacement in the figure.

70.] **Projection of a Displaced String of Constant Length.**

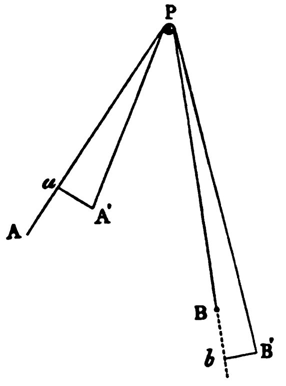

Fig. 76.

Let APB be a string which passes over a peg at P, and, the length of the string remaining the same, let the extremities, A and B, be slightly displaced to A' and B'. Let Aa and Bb be the projections of the displacements AA' and BB' on the original portions of the string (Fig. 76). Then $Aa = Bb$.

For $Pa = PA'.\cos aPA' = PA'$, as in the last Article.

Also $Pb = PB'$. Hence, since $PA' + PB' = PA + PB$, $Pa + Pb = PA + PB$, therefore $Aa = Bb$.

If in the last Article $l =$ the length of AB, and in the present, $l =$ length of the string, both results are expressed in the equation

$$\delta l = 0.$$

71.] **Virtual Work of the Tension of an Inelastic String.** In Fig. 76 suppose the peg to be smooth. Let A and B be two particles which are acted on by any forces which keep the system in equilibrium in the position indicated by the figure. Then if we consider the equilibrium of A alone, we may replace the string by a force $= T$ (the tension) acting in AP. Considering then a virtual displacement AA', the tension would furnish the term

$$T.Aa, \text{ or } -T.\delta r,$$

to A's equation of virtual work, the length PA being denoted by r. Similarly, considering the equilibrium of B, the tension would furnish to its equation of virtual work, for the virtual displacement BB', the term

$$-T.Bb, \text{ or } -T.\delta r',$$

r' denoting the length of PB.

If we combine the two equations by addition, the term contributed by the tension will be

$$-T(\delta r + \delta r'), \text{ or } -T.\delta l,$$

which = 0, since the particles A and B are imagined to be simultaneously displaced in such a manner that the length of the connecting string is constant. Hence—

For any small virtual displacement in which the length of a string is unaltered, the virtual work of its tension = 0.

In the same way, if, in Fig. 74, the rod AB, connecting two particles A and B, be subject to a tension, T, in the direction of its length, the virtual work of this tension for the displacement $A'B'$ will be

$$T\,.\,(Aa-Bb),\ \text{or}\ -T\,.\,\delta AB,$$

which = 0, because the length of AB is constant.

Hence—*The virtual work of the tension of a rod connecting two points whose mutual distance is unaltered in the virtual displacement is zero.*

72.] **Typical Expression for the Virtual Work of a Force. Example.**—We have seen (Art 65) that if a force, P, act on a particle, O, whose virtual displacement, OA, has a projection $= \delta p$ on the line of action of P *in the direction in which P acts*, the virtual work of P is

$$P\,.\,\delta p.$$

Generally, *if p denote the co-ordinate, referred to some fixed axis, of the point of application of a force, P, whose direction is perpendicular to the axis, the virtual work of the force is* $P\,.\,\delta p$, δp *being supposed to be a positive increment, and the co-ordinate being measured in the sense in which P acts.*

As an example, let us determine the relation between the two connected masses, P and P' (Fig. 77), which rest on two smooth inclined planes, of inclinations i and i'. Let y and y' denote the co-ordinates of the centres of gravity of the masses, referred to a horizontal plane through O. Then the equation of virtual work for the system, the displacements being imagined to be along the planes, is

$$P\,.\,\delta y+P'\,.\,\delta y' = 0. \qquad (1)$$

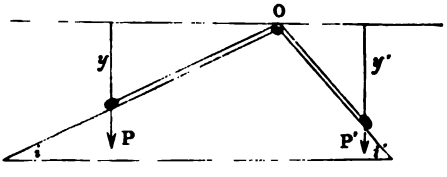

Fig. 77.

[Here it will be observed that the normal reactions do not enter, because the virtual displacements take place at right angles to them (see Art. 66); and the tension does not enter, since the virtual displacement does not alter the length of the string (see Art. 71)].

To this must be added the geometrical equation connecting y and y'. If l be the length of the string, we have, clearly,

$$\frac{y}{\sin i} + \frac{y'}{\sin i'} = l.$$

Differentiating this equation, we have

$$\frac{\delta y}{\sin i} + \frac{\delta y'}{\sin i'} = 0. \tag{2}$$

Hence, from (1) and (2),

$$P \sin i = P' \sin i',$$

an equation which is, of course, otherwise evident.

If the masses are connected as in example 16, p. 63, we have still the equation of virtual work,

$$P\delta y + Q\delta y' = 0, \tag{3}$$

y and y' denoting the vertical distances of P and Q in the figure of that example from a horizontal plane through C.

From (3) we can deduce the relation between P and Q in terms of the angles θ and ϕ given in example 16, p. 63. For, $y = h \cos a (\sin a \cot \theta + \cos a)$, with a similar value of y'; therefore

$$\delta y = -\frac{h \sin a \cos a}{\sin^2 \theta} \cdot \delta\theta;\ \delta y' = -\frac{h \sin \beta \cos \beta}{\sin^2 \phi} \cdot \delta\phi.$$

But $\dfrac{\cos a}{\sin \theta} + \dfrac{\cos \beta}{\sin \phi} = \dfrac{h}{l}$, $\therefore$ by differentiating this, we have $\dfrac{\cos a \cos \theta}{\sin^2 \theta} \cdot \delta\theta + \dfrac{\cos \beta \cos \phi}{\sin^2 \phi} \cdot \delta\phi = 0$, so that $\dfrac{\delta y}{\delta y'} = -\dfrac{\sin a \cos \phi}{\sin \beta \cos \theta}$, and this $= -\dfrac{Q}{P}$, by (3). Hence the relation between the masses before obtained.

73.] **Forces of Constraint.** When a particle is compelled to satisfy some geometrical condition—as, for instance, to rest on a given smooth surface, or to preserve a constant distance from some other particle—this condition is equivalent to the action of a certain force on the particle. If the particle is compelled to rest under given forces on a smooth inclined plane, we have seen that this condition may be removed if we produce, by any means, a force exactly equal to the normal reaction of the plane on the particle. In the same way, the connexion of the particle with another by means of a rigid rod may be severed if we produce on the particle the force which is actually impressed upon it by the rod.

Forces proceeding from geometrical connexions are sometimes called *geometrical forces*, or *forces of constraint*, and if these forces are actually produced on the particle by other means, the conditions may be violated, and the particle considered absolutely free from constraint.

74.] Choice of Virtual Displacements. When two or more particles constituting a system are connected by rods or strings, and constrained to rest on given smooth curves or surfaces, there is an advantage, when seeking for the position of equilibrium, in choosing *such virtual displacements as do not violate any of these conditions ;* because, as we have seen, the tensions of the connecting rods or strings, and the reactions of the smooth curves or surfaces, will, for such virtual displacements, contribute nothing to the equation of virtual work of the system. Thus we get rid at once of all such unknown forces. Of course, any geometrical condition may be violated in a virtual displacement at the expense of bringing into the equation of virtual work the corresponding force of constraint.

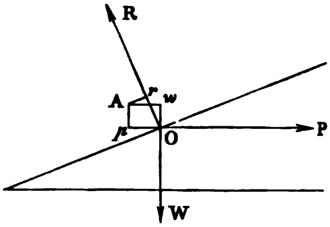

Fig. 78.

For example, if a particle, O (Fig. 78), is placed on a smooth plane whose inclination is i, and we wish to find the horizontal force, P, which will sustain it, the best displacement to choose is one along the plane, i. e. one which does not violate the geometrical condition, because, if this is chosen, the unknown reaction, R, will not appear in the equation of virtual work. But we shall still get a valid equation if we choose a virtual displacement, OA, which does violate the condition. This equation is

$$R \,.\, Or - P \,.\, Op - W \,.\, Ow = 0,$$

Or, Op, and Ow being the projections of OA on the directions of R, P, and W, respectively.

On the other hand, if we wish to determine R, without determining P, the best virtual displacement to choose is one at right angles to P, i. e. a vertical displacement, which does violate the geometrical condition.

In the typical expression, $P\delta p$, for the virtual work of a force the letter δ has been used to signify that the small displacement is any whatever; but it is usual in the Differential Calculus to denote small increments of the co-ordinates of a point on a curve or surface by the letter d, when we pass from the point to an adjacent one *which also lies on the curve or surface.* Hence in the following examples, in which such passage alone is contemplated, we shall denote small displacements on the curves considered by this letter.

Examples.

1. Two heavy particles, P and P' (Fig. 79), rest on the concave side of a smooth vertical circle, and are connected by a string passing over a smooth peg, A, at the extremity of the vertical diameter. If the particles are acted upon by two horizontal forces, F and F', proportional to the distances, PQ and $P'Q'$, of the particles from the vertical diameter, find the position of equilibrium by the principle of virtual work.

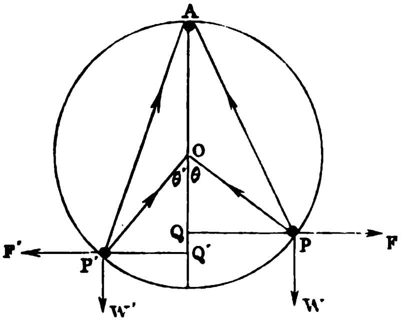

Fig. 79.

Let θ and θ' be the angles which the radii to P and P' make with the vertical; let the weights of the particles be W and W'; the radius of the circle $= a$, the length of the string $= l$, and the forces F and $F' = \mu . PQ$ and $\mu' . P'Q'$, respectively. Finally, let the distances PQ and $P'Q'$ be x and x', and let the vertical distances of P and P' below the horizontal diameter of the circle be y and y'.

Then, choosing virtual displacements of P and P' along the circle in such a manner that the length of the connecting string remains unaltered, we have

$$Wdy + W'dy' + Fdx + F'dx' = 0,$$

or

$$Wdy + W'dy' + \mu x . dx + \mu' x' . dx' = 0. \qquad (1)$$

Now $y = a\cos\theta, \quad y' = a\cos\theta', \quad x = a\sin\theta, \quad x' = a\sin\theta'.$

Hence (1) becomes

$$(W - \mu a\cos\theta)\sin\theta d\theta + (W' - \mu' a\cos\theta')\sin\theta' d\theta' = 0. \qquad (2)$$

Again, $AP = 2a\cos\frac{\theta}{2}$, $AP' = 2a\cos\frac{\theta'}{2}$.

Hence the geometrical equation is

$$\cos\frac{\theta}{2} + \cos\frac{\theta'}{2} = \frac{l}{2a}. \tag{3}$$

Differentiating this, we have

$$\sin\frac{\theta}{2}\cdot d\theta + \sin\frac{\theta'}{2}\cdot d\theta' = 0. \tag{4}$$

From (2) and (4), we have, therefore,

$$(W - \mu a\cos\theta)\cos\frac{\theta}{2} = (W' - \mu a\cos\theta')\cos\frac{\theta'}{2}. \tag{5}$$

The solution of the problem is contained in equations (3) and (5).

2. Two heavy particles, P and P', rest on two smooth curves in a vertical plane, and are connected by an inextensible string which passes over a smooth peg, A (Fig. 80), in the same plane. Prove that in the position of equilibrium, the centre of gravity of the particles is at the greatest or least height above the horizon that it can occupy consistently with the given conditions.

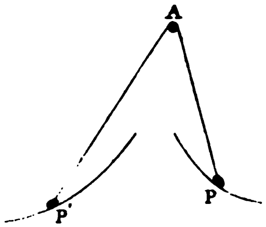

Fig. 80.

Let y and y' denote the vertical distances of P and P' from a horizontal line through A (or through any other fixed point). Then, the displacement being made consistently with the geometrical conditions, we have

$$W dy + W' dy' = 0, \tag{1}$$

W and W' being the weights of P and P'.

Now, the depth of the centre of gravity is $\bar{y}$, where

$$\bar{y} = \frac{Wy + W'y'}{W + W'}. \tag{2}$$

Hence, differentiating (2),

$$(W + W')d\bar{y} = W dy + W' dy' = 0; \tag{3}$$

and $\bar{y}$ is therefore a maximum or minimum.

If equation (3) holds in all positions of the particles, they will rest in all positions, and their centre of gravity is at a constant height.

3. Solve example 12, p. 19, by Virtual Work.

(Choose a displacement of O along the diameter of the circle.)

4. If the normals at P and P' (Fig. 80) meet the vertical line through A in n and n', prove that in the position of equilibrium

$$W\frac{AP}{An} = W'\frac{AP'}{An'},$$

a result which is at once evident from the *triangle of forces*.

5. If the particle P hang freely, find the curve on which P' will rest in all positions of the system.

Ans. A conic having A for focus.

6. If P and P' rest in all positions, and if the curve on which P' rests is given, find that on which P rests.

Ans. Let the horizontal line through A be taken as axis of x, $l =$ the length of the string, $y' = f(AP')$ be the equation of the given curve, and $Wy + W'y' = k$; then the equation of the other curve will be

$$Wy = k - W'f(l-r), \quad \text{or} \quad r = \phi(y),$$

where $r = AP$.

7. A particle is attracted towards two fixed points by two constant forces: find the curve on which it will rest in all positions.

Ans. A Cartesian oval.

8. A particle is acted upon by forces emanating from a given number of fixed points and proportional, respectively, to the distances of the particle from the fixed points; find (by Virtual Work) the surface on which the particle will rest in all positions.

Ans. A Sphere. [See also p. 22.]

9. Show from p. 94 that the two systems of curves obtained by varying C and C' in the equations

$$\begin{aligned} m_1 \log r_1 + m_2 \log r_2 + m_3 \log r_3 + \dots &= C, \\ m_1\theta_1 + m_2\theta_2 + m_3\theta_3 + \dots &= C', \end{aligned}$$

cut each other orthogonally.

10. A small ring is carried on a smooth wire bent into the shape of the magnetic curve; what is the relation between two forces directed towards the poles if they keep the ring in equilibrium?

Ans. If the forces towards N and S are P and P', both attractive or both repulsive, and if $\angle NPS = \omega$, while $NP = r$, $SP = r'$,

$$P(r^2 - r'^2 \cos\omega) + P'(r'^2 - r^2 \cos\omega) = 0.$$

[The student is recommended to solve some of the examples in pp. 64–66 by the Principle of Virtual Work.]

CHAPTER V.

COMPOSITION AND RESOLUTION OF FORCES ACTING IN ONE PLANE ON A RIGID BODY.

75.] **Resultant of two Parallel Forces.** Let two parallel forces, P and Q (Fig. 81), act at points A and B, in the same sense, on a rigid body. It is required to find the resultant of the forces P and Q.

At A and B introduce any two equal and opposite forces, F. The introduction of these forces will not disturb the action of P and Q, since, the body being indeformable (see Art. 22), the force F at A may be supposed to be transferred to B, at which point it would be directly opposed to the other force, F. Compound P and F at A into a single force, R, and compound Q and F at B into a single force, S. Then let R and S be supposed to act at O, the point of intersection of their lines of action, which is to be considered as rigidly attached to the body. At this point let them be resolved into their components, P, F, and Q, F, respectively. The forces F at O destroy each other, and the components P and Q are superposed in a right line, OG, parallel to their lines of action at A and B. The *magnitude* of the resultant is, therefore, $P+Q$. To find the point, G, in which its line of action intersects AB, let the extremities of P and R (acting at A) be joined. Then the triangle PAR is evidently similar to the triangle GOA; there-

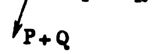

Fig. 81.

fore $\frac{P}{F} = \frac{OG}{GA}$. Similarly, $\frac{Q}{F} = \frac{OG}{GB}$; therefore, by division, $\frac{P}{Q} = \frac{GB}{GA}$. Hence—

The resultant of two parallel forces acting in the same sense at the extremities of a given line divides this line internally into two segments in such a way that each segment is inversely proportional to the force acting at its extremity.

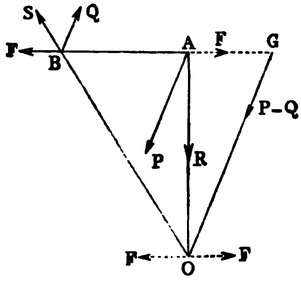

Fig. 82.

Suppose, now, that the parallel forces, P and Q, act in opposite senses. At A and B (Fig. 82), let two equal and opposite forces, F, be introduced, as before; and let R, the resultant of P and F, and S, the resultant of Q and F, be transferred to O, their point of intersection, supposed to be rigidly attached to the body. If at O the forces R and S are decomposed into their original components, it is clear that the system will reduce to a force, P, acting in the direction GO, parallel to the direction of P and Q, and a force, Q, acting in the direction OG. Hence the resultant is a force $=P-Q$ acting in the line GO. To determine the point G, we have, from the similar triangles, PAR and OGA,

$$\frac{P}{F} = \frac{OG}{GA}; \text{ also we have } \frac{Q}{F} = \frac{OG}{GB}; \text{ therefore } \frac{P}{Q} = \frac{GB}{GA}.$$

Hence—

The resultant of two parallel forces acting on a rigid body in opposite senses at the extremities of a given line cuts this line externally into two segments, in such a way that each segment is inversely proportional to the force acting at its extremity.

DEF.—The *segments* of a right line, AB, made by a point G in it or its production, are the distances, GA and GB, of the point G, from the extremities A and B of the given line, whether G is on AB, or on AB produced.

In both cases we have the equation

$$P \times GA = Q \times GB.$$

Hence we have, evidently, the theorem—

If from a point on the resultant of two parallel forces a right *line be drawn meeting the forces, whether perpendicularly or not, the products obtained by multiplying each force by its distance from the resultant, measured along the arbitrary line, are equal.*

76.] **Composition of Parallel Forces deduced directly from that of Concurrent Forces.** Let two forces, P and Q (Fig. 83), act, in inclined directions, at two points, A and B, of a rigid body. Let O be the point in which their lines of action meet, and measure off Om and On proportional to P and Q respectively. Then, completing the parallelogram $Omrn$, the diagonal, Or, represents the resultant of P and Q in magnitude and direction. Let G be the point in which Or meets AB. Then we have

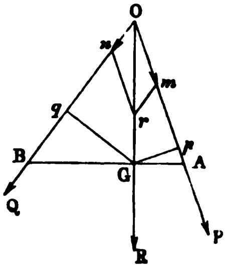

Fig. 83.

$$\frac{P}{Q} = \frac{Om}{mr} = \frac{\sin r\,On}{\sin r\,Om},$$

From G let fall perpendiculars, Gp and Gq, on P and Q. Then $\sin r\,On = \frac{Gq}{GO}$, and $\sin rOm = \frac{Gp}{GO}$; therefore

$$\frac{P}{Q} = \frac{Gq}{Gp}. \tag{1}$$

Again, if R is the resultant of P and Q, we have

$$\frac{R}{P} = \frac{Or}{Om} = \frac{\sin n\,Om}{\sin n\,Or},$$

or

$$\frac{R}{P} = \frac{\text{perp. from } B \text{ on } P}{\text{perp. from } B \text{ on } R}. \tag{2}$$

Now, if P and Q are parallel, R becomes parallel to P and Q, and we shall evidently have $\frac{Gq}{Gp} = \frac{GB}{GA}$; hence (1) gives for parallel forces

$$\frac{P}{Q} = \frac{GB}{GA};$$

and (2) gives, since R is parallel to P and Q,

$$\frac{R}{P} = \frac{BA}{BG} = \frac{BG + GA}{BG} = 1 + \frac{Q}{P},$$

$$\therefore \quad R = P + Q.$$

A similar demonstration holds when P and Q act in opposite senses.

77.] **Construction for the Resultant of two Parallel Forces.** If the lines AP and BQ (Figs. 84 and 85) represent in magnitudes and lines of action two parallel forces, the student will easily prove the following construction for the resultant:—

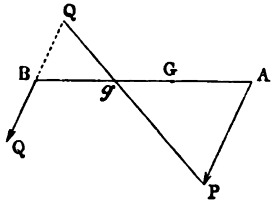

Fig. 84.

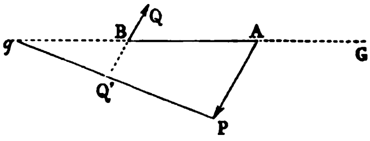

Fig. 85.

Draw BQ' equal and opposite to BQ, and draw PQ', meeting AB in g. Then measure off $AG = Bg$. G is a point on the resultant. Through G draw an indefinite right line parallel to P and Q, and from A and P draw parallels to PQ' and AB, respectively. These lines will intercept on the line through G a length $= P + Q =$ resultant.

78.] **Moment of a Force with respect to a Point.** Let a force, P (Fig. 86), act on a rigid body in the plane of the paper, and let an axis perpendicular to this plane pass through the body at any point, O. It is clear, then, that the effect of the force will be to turn the body round this axis, (the axis being supposed to be fixed,) and the rotatory effect will depend on two things—firstly, the magnitude of the force, P, and, secondly, the perpendicular distance, p, of P from O. If P passes through O, it is evident that no rotation of the body round O can take place, whatever be the magnitude of P; while if P vanishes, no rotation will take place, however great p may be. The product

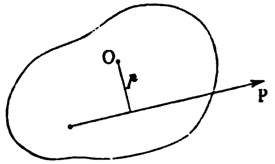

Fig. 86.

$$P.p$$

is called the *Moment* of the force about the axis through O. Its precise connection with the rotational motion which would be produced in the body, if this body were free to revolve round the axis through O, cannot be seen by the student until he studies Kinetics; but it is not at all necessary to enquire into this matter here. Even if a Moment had no such kinetical reference, its discussion in the theory of the composition, &c. of forces would be necessary—as will be at once seen.

When all the forces under consideration act in one plane, we may speak of the *point*, O, in which the axis of *Moments* meets this plane, instead of the axis itself. We shall therefore define the *Moment*, with respect to a point, of a force acting on a body to be *the product of the force and the perpendicular let fall on its line of action from the point.*

If the unit of force is a pound weight and the unit of length a foot, the unit of Moment will obviously be a *foot-pound weight.*

79.] **Moments of different Signs.** *If two forces tend to produce rotations of a body in opposite senses round a point, their moments with respect to this point are affected with opposite signs.* Thus (Fig. 87), the force P tends to turn the body round O in a sense opposite to that of watch-hand rotation, while Q tends to turn it in the opposite sense. If, then, the former rotation is considered positive, the algebraic sum of the moments of P and Q round O is

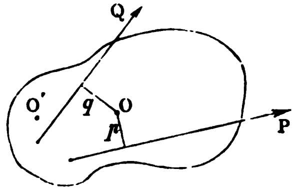

Fig. 87.

$$P.p-Q.q,$$

p and q being the perpendiculars from O on P and Q.

Round the point O' both forces would produce rotation in the same sense, and therefore the algebraic sum of their moments with respect to this point is

$$P.p'+Q.q',$$

p' and q' being the perpendiculars from O' on P and Q, respectively.

In future we shall usually speak of the *sum* of the moments, instead of the *algebraic sum* of the moments, of forces with respect to a point, as we shall suppose the moment of each force to be affected with its proper sign, in accordance with the rule given at the beginning of this Article.

80.] **Case of two Equal and Opposite Parallel Forces.** If the forces P and Q in Art. 75, Fig. 82, are equal, the equation

$$P \times GA = Q \times GB$$

gives $GA = GB$, or $\frac{GA}{GB} = 1$, an equation which is true only when G is at infinity on AB. Also the resultant of the forces being equal to their difference, is equal to zero. Two equal and opposite parallel forces acting on a rigid body constitute what is called a *Couple.*

Hence, regarded in one way, a couple is equivalent to a zero force acting along a line at infinity, its point of application being, however, supposed to be rigidly connected with the body. Observe, then, that, though the force is of zero magnitude, it has an infinitely long 'lever-arm' with respect to any point at a finite distance; so that it must not be rejected as something without meaning.

We now proceed to show that, regarding couples in a different way, they possess remarkable properties.

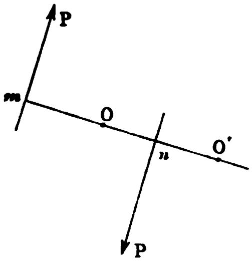

Fig. 88.

THEOREM I. *Two equal and opposite parallel forces have a constant moment with respect to all points in their plane.*—Let O (Fig. 88) be any point in the plane of two equal and opposite parallel forces, P, and let fall the perpendiculars Om and On on their lines of action. Then, if O is inside the lines of action of the forces, these forces tend to produce rotation round O in the same sense, and therefore the sum of their moments is equal to

$$P(Om + On), \quad \text{or} \quad P \times mn.$$

If the point chosen is O', the sum of the moments is evidently

$$P(O'm - O'n), \quad \text{or} \quad P \times mn,$$

which is the same as before.

The perpendicular distance between the two forces of a couple is called the *Arm* of the couple.

The *Moment* of a couple is the product of the arm and one of the forces.

The *Axis* of a couple is a right line drawn anywhere perpendicular to the plane of the couple, and in a particular sense, its length being proportional to the moment of the couple. The sense of the axis is determined thus:—imagine a watch placed in the plane in which several couples act. Then let the axes of those couples which tend to produce rotation in the direction opposed to that of the rotation of the hands be drawn upwards through the face of the watch, and the axes of those which tend to produce the contrary rotation be drawn downwards. The extremities of these axes may very properly be marked with arrow-heads as in the case of lines representing forces.

THEOREM II. *The effect of a couple on a rigid body is not altered if the arm is turned through any angle round one extremity.*

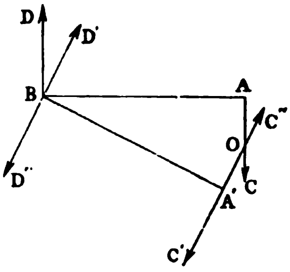

Fig. 89.

Let AC and BD (Fig. 89) be a couple whose arm is AB, and let the arm turn round B into the position BA'. At A' introduce two equal and opposite forces, $A'C'$ and $A'C''$, each of which is equal to one of the forces, P, of the given couple, and perpendicular to BA'. At B introduce two equal and opposite forces, BD' and BD'', perpendicular to BA', each force being equal to AC or P. The effect of the given couple is, of course, unaltered by the introduction of these forces. Now the forces BD and BD'' may be replaced by their resultant, $2P\cos\frac{DBD''}{2}$, or $2P\sin\frac{ABA'}{2}$, which acts in the bisector, BO, of the angle DBD''; and the forces AC and $A'C''$ may be replaced by their resultant, $2P\cos\frac{COC''}{2}$, or $2P\sin\frac{ABA'}{2}$, which also acts in the line BO in a sense opposed to the previous resultant. Hence the forces BD, BD'', AC, and $A'C''$, are a null system. There remain, then, the forces BD' and $A'C'$ which form a couple whose arm is BA'. Hence the couple of forces P acting at A and B may be replaced by a couple of forces P acting at the extremities of an arm of length equal to AB having one extremity common with AB.

THEOREM III. *The effect of a couple on a rigid body is not altered if the arm is moved parallel to itself anywhere in the plane of the Couple.*

Let two forces, AC and BD, each equal to P (Fig. 90), act with arm AB, and draw $A'B'$ equal and parallel to AB in the plane of the couple, this line $A'B'$ being supposed to be rigidly connected with the body. At A' and B' introduce, perpendicularly to $A'B'$, four forces $A'C'$, $A'C''$, $B'D'$, and $B'D''$, each equal to P. This does not alter the effect of the given couple. Now since AB and $A'B'$ are equal and parallel, the lines AB' and BA', being the diagonals of the parallelogram $ABB'A'$, bisect each other in the point O, suppose. Replace the forces BD and $A'C''$ by their resultant, $2P$, which acts at O parallel to BD; and replace the forces AC and $B'D''$ by their resultant, $2P$, which also acts at O in a sense opposite to the previous resultant. These two resultants destroy each other, and therefore the forces BD, AC, $B'D''$, and $A'C''$, constituting a null system, may be removed. There remain the forces, $A'C'$ and $B'D'$, which constitute a couple whose arm is $A'B'$. Therefore, &c.

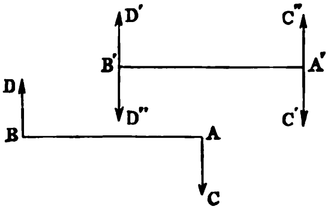

Fig. 90.

Theorem IV. *The effect of a Couple on a rigid body is not altered if the Couple is changed into another having the same moment, the arms of the Couples being in the same line and having a common extremity.*

Let the given couple be AC and BD (Fig. 91), each equal to P. Produce BA to A' so that $\frac{BA}{BA'} = \frac{Q}{P}$, and at A' and B introduce equal and opposite forces $A'C'$ and $A'C''$, BD' and BD'', the magnitude of each of these forces being Q. Now the forces AC and $A'C''$ give a resultant $= P - Q$ at B (Art. 75) along the line BD''; and this force added to BD'' gives a force $= P$ which destroys BD. Hence there remain the forces $A'C'$ and BD', which form a couple whose moment is equal to that of AC and BD, since (by construction)

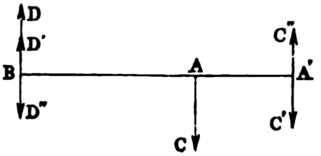

Fig. 91.

$$P.BA = Q.BA'.$$

Therefore, &c.

Theorem V. *A Couple acting on a rigid body may be replaced by any other Couple in the same plane if the moments of the Couples are the same in magnitude and sign.*

Let (P, P) and (Q, Q) (Fig. 92) be two Couples in the same plane, having the same moment, and tending to produce rotation in the same sense; then (P, P) may be transformed into (Q, Q). For, we can first turn the arm AB round B until it is parallel to $B'A'$ (Theorem II); then we can lengthen it until it becomes equal to $B'A'$, changing, at the same time, the forces P into forces Q (Theorem IV); and finally, we can move it into the position $B'A'$ (Theorem III).

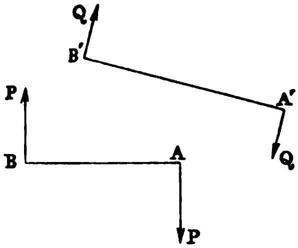

Fig. 92.

The sign of the moment of a couple is indicated by the sense in which the axis is drawn, as has been already explained. Axes drawn upwards through the face of the watch are then considered positive, and axes drawn downwards are negative.

Theorem V can be proved directly, without the aid of the previous propositions.

From the foregoing Theorems it is clear that the addition of coplanar couples is effected by adding their axes, regard being had to the signs of the axes.

THEOREM VI. *A force and a couple acting in the same plane on a rigid body are equivalent to a single force.*

Let the force be F and the couple (P, a)—that is, P is the magnitude of each force in the couple whose arm is a. Then (Theorem IV) the couple $(P, a) =$ the couple $\left(F, \frac{aP}{F}\right)$. Let the latter be moved until one of its forces acts in the same line as the given force F, but in the opposite sense. The given force F will then be destroyed, and there will remain a force F acting in the same direction as the given one and at a perpendicular distance $= \frac{aP}{F}$ from it.

This Theorem is equivalent to the statement—*A force and a couple acting in the same plane cannot produce equilibrium.*

THEOREM VII. *A force acting on a rigid body at any point A may be replaced by an equal force acting in the same direction at any other point B together with a couple whose moment is the moment of the original force about B.*

This important proposition is easily demonstrated.

THEOREM VIII. *The resultant of any number of Coplanar Couples is a couple whose moment is equal to the sum (with the proper signs) of the moments of the given couples.*

For, let the component couples have moments $L, M, N, \ldots$ and let each of them be changed into a couple, having the same right line AB (whose length is x) for arm. Then (Theorem IV), the couple L will give rise to a force $\frac{L}{x}$ at A, and an equal force in opposite sense at B. Hence at A we shall have the force $\frac{L+M+N+\ldots}{x}$ and an equal and opposite force at B. Thus we have a couple whose moment is the product of this force by the arm x; i.e., its moment is $L+M+N+\ldots$, or the sum of the given moments.

81.] **Geometrical representation of the Moment of a Force with respect to a Point.** Let the line AB (Fig. 93) represent a force in magnitude and direction, and let it be required to represent its moment with respect to a point O. If $p =$ the perpendicular from O on AB, the moment is $AB \times p$. Now this is double the area of the triangle AOB. Hence *the moment of a force with respect to a point is geometrically represented by double the area of the triangle whose base is the line representing the force in magnitude and line of action, and whose vertex is the given point.*

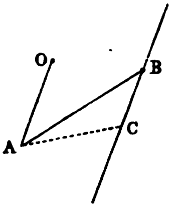

Fig. 93.

Draw AO, and from the other extremity, B, of the given force draw an indefinite right line, BC, parallel to AO. Join A to any point, C, of this line. Then the area of the triangle $AOB =$ the area of the triangle AOC, since these triangles have the same base and are between the same parallels. Consequently the moment of a force represented by AB about $O =$ the moment of a force represented by AC about O, wherever C be taken on the indefinite line through B.

82.] **Varignon's Theorem of Moments.** *The sum of the moments of two forces with respect to any point in their plane is equal to the moment of their resultant with respect to the point.*

Let AP and AQ (Fig. 94) represent two forces whose resultant is AR, and let O be the point about which moments are taken. Draw AO, and draw PC and QD parallel to it.

By the last Article the moment of AP about $O =$ the moment of AC about O, and the moment of $AQ =$ the moment of AD; therefore the sum of the moments of AP and AQ about $O =$ the sum of the moments of AC and AD about $O =$ the moment of the sum of AC and AD (since AC and AD are forces acting in the same line); but, by equal triangles AC is evidently $= DR$; therefore the sum of the moments $=$ the moment of $AR =$ the moment of the resultant. Q.E.D.

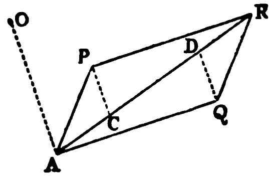

Fig. 94.

This result can also be very simply obtained by expressing the fact that the sum of the components of AP and AQ along a perpendicular to $AO =$ the component of AR in this direction, and multiplying both sides of the equation by the length OA.

The student will find no difficulty in considering the case in which O is between AP and AQ, observing that in this case their moments are opposed, and that in the new figure AR will be equal to $AD \backsim AC$.

Of course it follows that the sum of the moments (with their proper signs) of any number of coplanar forces with respect to any point in their plane is equal to the moment of their resultant with respect to this point; for the forces may be replaced in pairs by their resultants, &c. It also follows that the sum of the moments of the forces about any point on the line of action of the resultant is equal to zero.

83.] **Varignon's Theorem of Moments for Parallel Forces.** *The sum of the moments of two parallel forces about any point is equal to the moment of their resultant about the point.*

Let the forces be P and Q (Fig. 95) and let O be the point about which moments are to be taken. From O let fall perpendiculars OA, OB, and OG on the lines of action of P, Q, and their resultant, R, and let the forces be applied at the points A, B, and G, respectively.

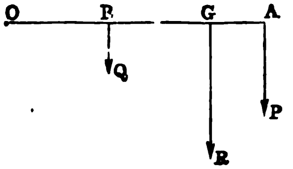

Fig. 95.

Then, moment of

$$P \text{ about } O = P \,.\, OA = P(OG + GA);$$

and moment of Q about $O = Q \,.\, OB = Q(OG - GB)$;

therefore, by addition, the sum of the moments

$$= (P+Q) \,.\, OG + P \,.\, GA - Q \,.\, GB.$$

But $P \,.\, GA = Q \,.\, GB$; therefore the sum of the moments

$$= (P+Q) \,.\, OG = R \,.\, OG.$$

A similar proof holds when P and Q act in opposite senses, and also when O is between the lines of action of P and Q.

It follows that *the sum of the moments (with their proper signs) of any number of coplanar parallel forces with respect to a point in their plane is equal to the moment of their resultant with respect to the point.*

84.] **Determination of the Resultant of Coplanar Forces.** Any system of coplanar forces, P_1, P_2, P_3, ... acting on a rigid body must either have a single resultant or reduce to a couple; for, since their lines of action all lie in one plane, we can produce the lines of action of P_1 and P_2 to meet, and replace them by their resultant, R_{12}; and in the same way we can compound this force and P_3 into a force R_{123}; and this again with P_4; and so on; so that finally we must obtain a single force, R, with finite magnitude and position, unless the last two to be compounded happen to be two equal parallel forces in opposite senses, in which case the system is equivalent to a couple.

Now the *magnitude* and *direction* (but not actual *line of action*) of the resultant, R, are at once obtained from the fact that its component along any line must be equal to the sum of the components of all the given forces along the line. Suppose then that we take *any* two rectangular directions, Ox, Oy, in the plane of the forces; let R make the angle θ with Ox, and let the components of P_1, P_2, P_3, ... along Ox be X_1, X_2, X_3, ... and along Oy be Y_1, Y_2, Y_3, ...; then

$$R\cos\theta = X_1 + X_2 + X_3 + \ldots, \text{ or } = \Sigma X, \qquad (1)$$

$$R\sin\theta = Y_1 + Y_2 + Y_3 + \ldots, \text{ or } = \Sigma Y. \qquad (2)$$

Hence R is represented in magnitude and direction by the diagonal of the rectangle whose sides are ΣX and ΣY; i.e.,

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}, \tag{3}$$

and
$$\tan\theta = \frac{\Sigma Y}{\Sigma X}. \tag{4}$$

These equations do not, of course, determine the actual line of action of R; this latter is to be determined by the principle of moments. Take, then, any point whatever, A, in the plane of the forces, and calculate the algebraic sum of the moments of the forces about this point. Suppose this to be M.

By what has been proved in the last two articles, M is the moment of the resultant, R, about A; so that we have merely to draw from A a line perpendicular to the already determined *direction* of R and take such a length, p, on this perpendicular at one side or the other of A, according as M is clockwise or counter-clockwise, that

$$R \times p = M. \tag{5}$$

This completes the determination of the resultant—sometimes called the *resultant of translation* of the forces.

In any specified system of forces we are to be guided in our choice of the point, A, round which to take moments by special considerations of simplicity—for example, if several of the forces act in lines passing through A.

85.] **Particular case in which the Resultant of Translation vanishes.** When forces applied to a particle have no resultant of translation, their whole effect is null. It is not necessarily so, however, if they are applied to a body of finite dimensions. For example—

If the forces acting upon a rigid body form by their magnitudes and lines of action the sides of a closed polygon taken in order, their resultant of translation vanishes, and they have a constant moment with respect to all points in their plane—i. e., they reduce to a couple.

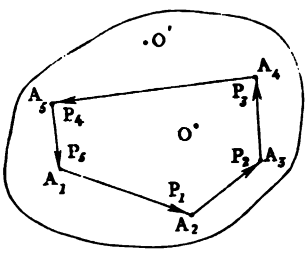

Fig. 96.

Let forces P_1, P_2, P_3, ... (Fig. 96) act at points, $A_1, A_2, A_3, \ldots$ in a body in one plane, and let these forces be represented in magnitudes and lines of action by the sides of the polygon formed by their points of application.

Now since the sum of the projections of the sides of this polygon on any arbitrary line = 0, the forces have no resultant of translation.

Let O be any point inside the polygon, and take the sum of the moments of the forces round it. If the perpendiculars from O on the sides A_1A_2, A_2A_3, ..., be p_1, p_2, ..., the sum of the moments will be

$$P_1p_1+P_2p_2+P_3p_3+\ \dots\text{, or } G\text{, suppose.}$$

And since P_1, P_2, ... are equal to the sides of the polygon, G is evidently = 2 . area of polygon. This is constant for all points inside the polygon.

Now if we take the sum of the moments round any external point, O', we shall have

$$P_1p_1+P_2p_2+P_3p_3-P_4p_4+P_5p_5,$$

since P_4 turns the body round O' in a sense opposite to that in which the other forces turn it. But this sum is equal to

$$2\,(A_1O'A_2+A_2O'A_3+A_3O'A_4-A_4O'A_5+A_5O'A_1),$$

and this is again equal to 2 . area of polygon.

Hence for all points in the plane, the sum of the moments, G, is constant.

Example of the determination of R.

$ABCD$ is a rectangle; $AB = 60$ inches, $AD = 45$ inches; in DC are taken two points, P, Q, such that $DP = 20$, $QC = 20$; and M is the middle point of AB. Find the resultant of the following forces (all measured in pounds' weight, suppose) represented by arrows in the figure: 10 in CD, 10 in DB, 18 at M perpendicular to AB, 20 at P making 45° with DC, 12 at Q making 30° with DC, and 16 at A making 30° with AD.

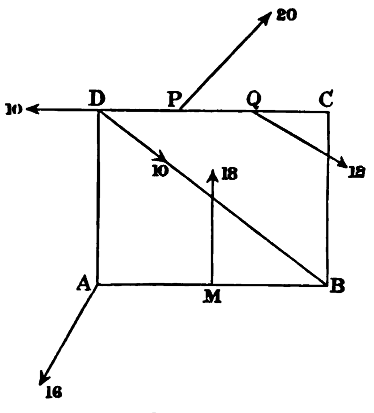

Fig. 97.

Here, taking the components parallel to AB, we have

$$\Sigma X = 14.534 \text{ in sense } AB,$$

and the components parallel to AD give

$$\Sigma Y = 6.286 \text{ in sense } AD.$$

Hence the resultant $R = 15.835$, and its direction is that of a line making $\tan^{-1}\dfrac{6.286}{14.534}$, or $\tan^{-1}.432$, with AB.

Again, D is the simplest point for moments, and we find

$$M_D^{\circlearrowleft} = 222.84 \text{ inch-pounds' weight},$$

the arrow indicating that the moment is counter-clockwise. Hence the line of action of R cuts AD between A and D, and the length of the perpendicular from D on it is 14.07 inches.

Special conditions for the partial or complete determination of the resultant of a system of coplanar forces may sometimes be given.

The system of forces being unspecified, if we are merely given that the sum of their moments about a given point A in their plane is L—in a clockwise sense, suppose—we can infer nothing as to the magnitude and line of action of R; but if we are given that the sum of the moments about a given point A is L, and about another given point, B, is M, both in the same sense, we infer a point on the line of action of R, viz., the point which divides the line AB produced into two segments in the ratio $L : M$; for if p and q are the perpendiculars from A and B on R, we have

$$R.p = L, \quad R.q = M,$$

therefore $p : q = L : M$, and it is obvious that R must pass through the point above named. If, in addition, we are given that the sum of the moments about a point C is N, in the same sense, suppose, R is completely determined, since its line of action must now also pass through the point dividing AC externally in the ratio $L : N$; and the perpendiculars on it from A, B, C are all known, and therefore the magnitude of R itself.

If L and M are of opposite senses, R passes through the point dividing AB internally in the ratio $L : M$.

EXAMPLES.

1. If the sums of the moments of a system of coplanar forces vanish for three points which are not in a right line, the system is in equilibrium.

2. If the sums of the moments round three points which are not in a right line are equal, and not each zero, the system is equivalent to a couple.

3. If the sums of the moments of a coplanar system round three points A, B, C are, respectively, proportional to the perpendiculars from these points on the opposite sides of the triangle ABC and have all the same sense, find the line of action of the resultant.

Ans. The line joining the points of intersection of the bisectors of the external angles with the opposite sides.

4. If L, M, N are the sums of the moments of a system of coplanar forces about three points A, B, C, show that the equation of the line of action of the resultant in trilinear co-ordinates referred to the triangle ABC is

$$aLx + bMy + cNz = 0,$$

where a, b, c are the lengths of the sides of the triangle.

5. Find the magnitude of the resultant in terms of L, M, N.

Ans.
$$R^2 = \frac{1}{4\Delta^2}\Sigma a^2(L-M)(L-N),$$

where Δ is the area of ABC, whose sides are a, b, c.

86.] **Conditions of Equilibrium of a Rigid Body acted upon by Coplanar Forces.** When a rigid body is acted upon by a system of coplanar forces, this system is reducible, generally, to a single force, but it may happen to reduce to a couple. The necessary and sufficient condition of equilibrium of the body evidently is that *this resultant of the forces shall be of zero magnitude without acting at infinity;* and the latter part of this condition will be guaranteed by the vanishing of the sum of the moments of the forces round any one point in their plane. Indeed, if the forces have no moment round any three points which do not lie in one right line, they cannot reduce either to a single force or to a couple; so that we may say that—*the necessary and sufficient conditions of equilibrium of a system of coplanar forces acting on a rigid body are that they give zero moment round each of any three points which do not lie in one right line.* (α)

Their equilibrium will also be guaranteed—i. e., they must have a zero resultant without reducing to a couple—if *the sums of their components, rectangular or oblique, vanish along any two particular lines in their plane, and also the sum of their moments about any one point vanishes.* (β)

For, if they have no total component along any direction (L) and also no total component along any other direction (L'), their resultant must be of zero magnitude, since its component along any direction is equal to the sum of the components of the forces in that direction; and if, in addition, their total moment round any one point vanishes, they cannot reduce to a couple.

When, therefore, a system of coplanar forces acting on a rigid body is in equilibrium we have the result that—

$$\left.\begin{array}{l}(1)\ \textit{the sum of their components along every line} = 0\,;\\(2)\ \textit{the sum of their moments about every point} = 0.\end{array}\right\}\qquad(\gamma)$$

The conditions which have just been enunciated are both necessary and *sufficient* for the equilibrium of any *rigid* material system to which the forces may be applied; but they are not sufficient for the equilibrium of a material system whose parts are not rigidly connected. The simplest way of seeing this is to remember that the whole process of compounding forces, two by two, into a single resultant is invalid and, indeed, without any meaning, unless they are all applied to one and the same rigid body. This was pointed out at the very outset (see p. 17). If a force, P, is applied at a point to a bar which is freely jointed to another bar to which a force P_2 is applied, the process of finding the resultant of P_1 and P_2 consists in producing their lines of action to meet in a point O, and then assuming that both P_1 and P_2 can be applied at O. But this assumes that O is at once rigidly connected with both bars, i.e., that the two bars are rigidly connected with each other; otherwise we cannot speak at all of the *resultant* of P_1 and P_2.

Let us imagine, then, any non-rigid, or deformable, system—a deformable framework of freely jointed bars, an elastic string, a fluid, &c.—at rest, and suppose that we are about to apply to this system a system of forces, $P_1, P_2, P_3, \ldots$ at specified points; what conditions must these forces satisfy in order that, when they are applied, the material system will continue to be at rest? Now, although we cannot completely answer the question without regard to the special features and structure of the material system, this much we *can* say, that—if we imagine the whole system to be suddenly made rigid, and if the forces will not keep this rigidified system at rest, they will, *à fortiori*, not suffice to keep the given deformable system at rest; for, rigidification always removes restrictions which hamper the action of the internal forces of the system, and renders these forces, therefore, more capable of preserving the equilibrium of the various connected parts against the action of external forces.

Hence the conditions expressed in (α), (β), (γ), though *necessary* for the equilibrium of forces applied to a deformable system, are not *sufficient*.

Analytical expression may be given to any of the forms (α), (β), (γ) which express the conditions of equilibrium. Thus, to express (β), let O (Fig. 98) be the point about which we take moments, and through O draw any two rectangular lines, Ox, Oy, which we may employ as axes of co-ordinates for the points of application of the various forces. Let P_1 act at A_1 and resolve the force P_1 into two components, X_1 and Y_1, parallel to Ox and Oy. Now (Art. 82), the moment of P_1 about O is equal to the sum of the moments of X_1 and Y_1 about O. But, if rotation opposite to that of a watch-hand is considered positive, the moment of Y_1 about O is $Y_1 . x_1$; and the moment of X_1 is $-X_1 . y_1$, where x_1 and y_1 are the co-ordinates of A_1 referred to the axes Ox and Oy. Hence the moment of P_1 about O is

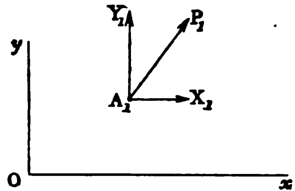

Fig 98.

$$Y_1 x_1 - X_1 y_1.$$

Adding together the moments of P_1, P_2, ..., we get the total moment

$$G = \Sigma(Yx - Xy). \tag{1}$$

If the sum of the components of the forces along Ox is denoted by ΣX, and the sum of the components along Oy by ΣY, the resultant of the forces (Art. 84) is given by the equation

$$R^2 = (\Sigma X)^2 + (\Sigma Y)^2. \tag{2}$$

Now, since for equilibrium we must have $R = 0$, and $G = 0$, the conditions, analytically expressed, are

$$\Sigma X = 0, \Sigma Y = 0, \tag{3}$$
$$\Sigma(Yx - Xy) = 0. \tag{4}$$

When the student comes to the study of the kinetics of a material system, he will see that if the external forces applied to it are such that $R = 0$ and $G = 0$, i. e., are such that if they acted on a *rigid* system, their resultant force and total moment round any axis both vanish, the centre of mass of the system is at rest (or in uniform motion in a fixed right line) and there is no resultant moment of mass-acceleration round any axis. This is true for all kinds of material systems, whether they are gases, liquids, deformable frameworks, natural solids, or rigid bodies.

Now the destruction of resultant linear and angular acceleration will, except in the case of rigid bodies, be quite consistent with the existence of motions of parts of the system among themselves, negative momenta cancelling positive. Hence, *whenever a system is capable of altering the relative positions of its parts, the complete equilibrium* of the system will require more than the vanishing of the resultants of translation and rotation of the forces applied to it. In fact, the internal forces will have to be taken into account. In rigid bodies the destruction of the above-mentioned motions will necessitate the destruction of all motion, and the conditions $X = 0$, $Y = 0$ are both necessary and sufficient. In these bodies there is no restriction placed on the internal forces, so that they are always capable of assuming such magnitudes and directions as will enable them to destroy the action of the external forces. On the contrary, in deformable bodies, there are restrictions placed on the internal forces so that they are not capable of preserving equilibrium against all systems of external forces. For example, in a freely jointed framework, the action between bar and bar must consist of a single force restricted to passing through the joint. This is the reason why two equal forces applied in opposite senses in the same line to two opposite sides of a set of parallel rulers will not hold them in equilibrium unless the rulers are placed in a certain configuration; and it is also the reason why two equal and directly opposed forces applied to the ends of a string, elastic or inelastic, will not hold it in equilibrium until it has assumed a certain state.

Hence also the necessity for considering the internal forces (pressures) in Hydrostatics.

Nevertheless, the conditions of equilibrium of *all* material systems whatever—natural solids, liquids, and gases—are completely expressed by the single principle that—*when the system has assumed its configuration of equilibrium, then for all imaginable small derangements of its parts the whole work which would thereby be done by all the acting forces, external and internal, is zero*—which is Lagrange's great principle of Virtual Work.

87. Reduction of a system of Forces. We have seen that a system of coplanar forces must ultimately reduce either to a single force or to a couple. It can be reduced to a single force acting at *any* point, O, supposed rigidly connected with the body, accompanied by a couple. For, let the system reduce to a single force, R, acting in a line (L). At O introduce two forces $(R, -R)$ parallel and equal to R and acting in opposite senses. Take R acting in the line (L) and $-R$ acting at O to make a couple; then, in addition to this couple, we have the force R at O. If p is the perpendicular from A on (L), the moment of the couple is

$$R \, . \, p,$$

i. e., it is the sum of the moments of the given forces about O. Thus the force at O is constant in magnitude and direction whatever point, O, is chosen, while the moment of the couple is variable.

The same thing is easily seen, less directly, by replacing each separate force $P_1, \dots$ by a force $P_1, \dots$ acting at O, accompanied by a couple (see Theorem VII. p. 111).

Thus, take the special case in which all the forces are parallel (Fig. 99).

Take any point O, and draw through it a right line, Oy, parallel to the forces. At O introduce two forces, P_1' and P_1'', each equal to P_1, these new forces being directly opposed to each other along Oy. Now, P_1 and P_1'' form a couple whose moment is $P_1 . p_1$, if p_1 is the perpendicular from O on the line $A_1 P_1$. Introducing in the same way, two forces, P_2' and P_2'', equal to P_2, directly opposite to each other along Oy, we have P_2 at A_2 replaced by a force P_2'' acting at O along Oy' and a couple whose moment is $-P_2 . p_2$, p_2 being the perpendicular from O on the line $A_2 P_2$. The — sign is attached to this couple because the couple (P_2', P_2) tends to produce rotation in a sense opposite to that in which the couple (P_1'', P_1) tends to produce rotation.

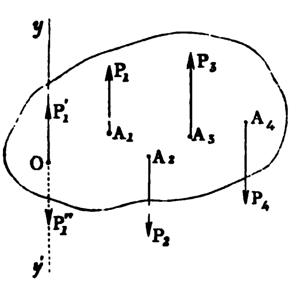

Fig. 99.

Proceeding in this way with all the forces in the above figure, we have the whole system of forces at A_1, A_2, A_3, A_4, ... equivalent to a single force,

$$P_1 - P_2 + P_3 - P_4 + \dots,$$

acting at O in the direction Oy, and a couple,

$$P_1 . p_1 - P_2 . p_2 + P_3 . p_3 - P_4 . p_4 + \dots,$$

tending to turn the body round O in a sense opposite to that of watch-hand rotation.

Denoting the resultant force by R, and the moment of the resultant couple by G, we have

$$R = \Sigma P, \qquad (1)$$

$$G = \Sigma (P . p). \qquad (2)$$

Again, let the forces act in any directions.

At O (Fig. 100) introduce two opposite forces, P_1' and P_1'', each equal and parallel to P_1. Let P_1 and P_1'' be considered as forming a couple. Then P_1 at A_1 is equivalent to P_1 acting at O, and a couple whose moment $= P_1 . p_1$. Replace P_2 at A_2 in the same way by P_2'' (or P_2) acting at O, and a couple (P_2, P_2') whose moment is $- P_2 . p_2$. Thus the whole system of forces will be replaced by forces P_1, P_2, P_3, P_4, .. , acting at O, and a number of couples

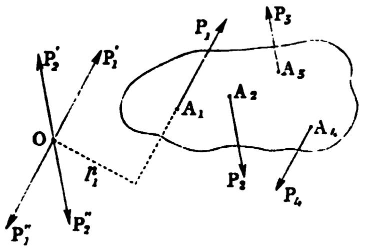

Fig. 100.

whose moments are $P_1 . p_1$, $-P_2 . p_2$, $P_3 . p_3$, $-P_4 . p_4, \ldots$ (the forces acting as in the above figure). The forces acting at O will have a single resultant, R, and the couples will form a single couple whose moment, G, is (Theor. VIII, Art. 80) the sum of the moments of the couples.

88.] **Equation of line of action of Resultant.** The equation of the line of action of the resultant of a system of coplanar forces is obtained with reference to any assumed axes of co-ordinates, Cartesian or Trilinear, by expressing the fact that the sum of the moments of the given forces round any point on the line of action of R is zero.

Thus, let Ox, Oy (Fig. 98) be any rectangular axes with respect to which the co-ordinates of the points of application of $P_1, P_2, \ldots$ are (x_1, y_1), $(x_2, y_2), \ldots$, and let (x, y) be the co-ordinates of any point in the plane. Then the moment of P_1 about (x, y) is evidently

$$Y_1(x_1-x)-X_1(y_1-y),$$

or

$$Y_1 x_1-X_1 y_1-xY_1+yX_1.$$

The moment of P_2 about (x, y) is

$$Y_2 x_2-X_2 y_2-xY_2+yX_2,$$

and similarly for the other forces. Hence the total moment, G', of the forces about the point is given by the equation

$$G' = G-x\Sigma Y+y\Sigma X, \tag{1}$$

where G is the sum of the moments of the forces about O and stands for $(Y_1 x_1-X_1 y_1) + (Y_2 x_2-X_2 y_2) + \ldots$.

If (x, y) is any point on R, $G' = 0$, so that the equation of the line of action of R is

$$x\Sigma Y-y\Sigma X = G. \tag{2}$$

The interpretation of equation (1) is obviously that—*The sum of the moments of a system of coplanar forces about any point, O, is equal to the sum of their moments about any other point, O′, plus the moment about O of their resultant of translation, supposed acting at O′*—a result which is evident from first principles.

Examples.

1. O is any point inside a triangle ABC; α, β, γ are the perpendiculars from O on the sides; if forces act along the sides a, b, c equal to $\frac{k}{\alpha}$, $\frac{k}{\beta}$, $\frac{k}{\gamma}$, respectively, and all in the same cyclical order, find the line of their resultant.

Take any point whose trilinear co-ordinates with reference to the triangle *ABC* are x, y, z; then the sum of the moments of the forces about it is $\frac{k}{\alpha}x + \frac{k}{\beta}y + \frac{k}{\gamma}z$. Hence the equation of the line of R is

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 0,$$

which is the axis of homology of *ABC* and the triangle *A′B′C′* formed by the points in which *AO*, *BO*, *CO* intersect the opposite sides.

If two of the forces, $\frac{k}{\alpha}$, $\frac{k}{\beta}$, have the same cyclical order while the third acts in the reverse order, the equation of R is

$$\frac{x}{\alpha} + \frac{y}{\beta} - \frac{z}{\gamma} = 0,$$

i.e., R acts in the line *A′B′*.

2. A triangular plate, *ABC* (Fig. 101), is acted upon at each angle by forces, along the two sides containing it, represented in magnitudes and lines of action by the distances between the angle and the feet of the perpendiculars let fall from the other two angles on these sides. Find the line of action of the resultant force.

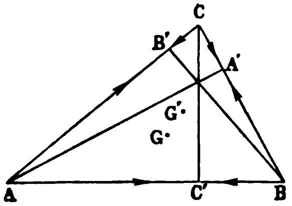

Fig. 101.

Let the perpendiculars let fall on the three sides, a, b, c, from any point, P, on the resultant be x, y, z, respectively, and let A', B', C' be the feet of the perpendiculars. Then the force in *AB* in the sense *AB* is $AC' - BC'$, or $b \cos A - a \cos B$. Hence the moment of this force about P is $z(b \cos A - a \cos B)$, and since the sum of the moments of all the forces (estimated in cyclical order) round P is $= 0$ (Art. 82), we have

$$x(c \cos B - b \cos C) + y(a \cos C - c \cos A) + z(b \cos A - a \cos B) = 0. \quad (1)$$

Now, one set of values of x, y, and z, which will satisfy this equation, is, evidently, a, b, c. Hence the resultant passes through the point the perpendiculars from which on the sides are proportional to a, b, c. This point is thus found:—Let G be the centre of gravity of the triangle; from A draw a line, AG', which makes $\angle CAG' = \angle BAG$, and from B draw a line, BG', which makes $\angle CBG' = \angle ABG$. These lines intersect in G', the required point which is called the isogonal conjugate of G, or the *Symmedian point.*

Again, another set of values of x, y, z, which will satisfy (1), is $\cos A$, $\cos B$, $\cos C$; and the resultant passes through the point whose perpendiculars on the sides are proportional to these quantities. This point is the centre of the circumscribed circle.

Hence the line of action of the resultant is known.

3. Show that the resultant of the system of forces in the last example is

$$\frac{4\Delta}{abc}\sqrt{a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2},$$

where Δ is the area of the triangle.

4. Forces P, Q, R, act along the sides of a triangle, ABC, and their resultant passes through the centres of the inscribed and circumscribed circles: prove that

$$\frac{P}{\cos B-\cos C}=\frac{Q}{\cos C-\cos A}=\frac{R}{\cos A-\cos B}$$

(Wolstenholme's Book of Mathematical Problems).

5. If from a point C on a circle there be drawn two chords CA, CB, prove that two forces inversely proportional to AC and BC acting along these lines both to, or both from, C, have a resultant passing through the Symmedian point of ABC and inversely proportional to the chord through this point.

6. If AD and BC (A, B, C, D being in cyclical order) are any two chords of a circle and two forces inversely proportional to them act along them in the senses AD, BC, their resultant passes through the intersection of AC and BD; but if they act in the senses AD, CB, it passes through the intersection of AB and DC.

7. If forces inversely proportional to the trilinear co-ordinates of any point on the circle circumscribing a triangle act, in cyclical order, along the corresponding sides, prove that their resultant will always pass through the isogonal conjugate of the centroid of the triangle.

8. Find the relation between three forces such that if they act along the sides of a triangle in cyclical order, their resultant will always touch the circumscribed circle.

Ans. If the forces are P, Q, R, and a, b, c are the sides,

$$\sqrt{aP}+\sqrt{bQ}+\sqrt{cR}=0,$$

so that the trilinear co-ordinates of any point on the inscribed conic which touches the sides at their middle points will represent all such force systems.

9. Find the relation between the forces if their resultant always touches the inscribed circle.

Ans. $$\frac{\cos^2\frac{1}{2}A}{P}+\frac{\cos^2\frac{1}{2}B}{Q}+\frac{\cos^2\frac{1}{2}C}{R}=0,$$

so that they can be represented by the trilinear co-ordinates of any point on a certain conic circumscribing the triangle.

(These results can be obtained purely by the principle of moments, Art. 82.)

10. Generally, if the resultant of forces acting in cyclical order

along the sides of a triangle touches any assigned curve, the relation between the forces is expressed by the tangential equation of the curve in trilinear co-ordinates.

89.] **Centre of Parallel Forces. Theorem.** *If any number of parallel forces,* $P_1, P_2, P_3, \ldots P_n$, *act in one plane at points* $A_1, A_2, A_3, \ldots A_n$, *their resultant passes through a fixed point if all the forces are turned in the same sense round their points of application through an arbitrary but common angle.*

The point, g_1 (Fig. 102), of application of the resultant of P_1 and P_2 has been determined (Art. 75) by dividing the line $A_1 A_2$ so that

$$\frac{A_1 g_1}{A_2 g_1} = \frac{P_2}{P_1},$$

on the supposition that the forces P_1 and P_2 are parallel, but no assumption has been made as to their common direction. Hence g_1 will be a point on their resultant in whatever direction they act, and the force at this point is $P_1 + P_2$. The point of application of the resultant of P_1, P_2, and P_3, is determined by joining g_1 to A_3, and dividing it in g_2, so that

$$\frac{g_1 g_2}{A_3 g_2} = \frac{\text{force at } A_3}{\text{force at } g_1} = \frac{P_3}{P_1 + P_2},$$

and the force at g_2 is $P_1 + P_2 + P_3$. Similarly, the point of application of the resultant of P_1, P_2, P_3, and P_4, is a point, G, on $g_2 A_4$, such that

$$\frac{g_2 G}{A_4 G} = \frac{P_4}{P_1 + P_2 + P_3},$$

and the force at $G = P_1 + P_2 + P_3 + P_4$.

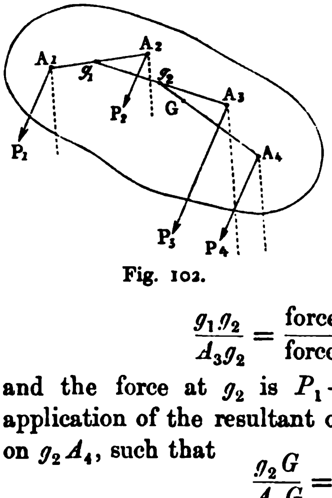

Fig. 102.

We thus see that the point, G, of application of the resultant of the system is determined by dividing the lines $g_1 A_3, g_2 A_4, \ldots$ in certain ratios which depend simply on the *magnitudes*, and not on the common *direction*, of the forces at $A_1, A_2, A_3, \ldots$. The theorem is, therefore, evident.

Of course no one point on the line of action of a force which acts on an indeformable body has a special right to be called *the* point of application of the force; and if we are given a parallel system, $P_1, P_2, P_3, \ldots$ with a fixed direction, *a* point on their resultant can be found by taking *any* points, B_1, B_2, B_3, ... on their respective lines of action and going through the above process; but the point so determined on the resultant is not one through which this resultant would continue to pass if the common direction of the parallel system were changed in such a way that the forces, respectively, revolved round a different system of points, A_1, A_2, A_3, ... If we contemplate such a revolution round these points, the point G is appropriately termed *the* centre of the system.

90.] Centre of Mean Position. Let there be any number of points, A_1, A_2, A_3, ... (Fig. 103), in one plane, and let the line, $A_1 A_2$, be divided at g_1 so that

$$\frac{g_1 A_2}{g_1 A_1} = \frac{m_1}{m_2};$$

let $g_1 A_3$ be divided at g_2, so that

$$\frac{g_2 A_3}{g_2 g_1} = \frac{m_1 + m_2}{m_3};$$

let $g_2 A_4$ be divided at g_3, so that

$$\frac{g_3 A_4}{g_3 g_2} = \frac{m_1 + m_2 + m_3}{m_4};$$

and so on, until by a final construction we arrive at a point G. It is required to express the distance of G from an arbitrary line, L, in the plane of the points in terms of the distances, $z_1, z_2, z_3, \ldots$ of A_1, A_2, A_3, ... from this line.*

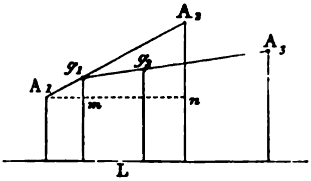

Fig. 103.

Draw $A_1 mn$ parallel to L. Then

$$\frac{g_1 m}{A_2 n} = \frac{A_1 g_1}{A_1 A_2} = \frac{m_2}{m_1 + m_2},$$

$$\therefore \quad g_1 m = \frac{m_2}{m_1 + m_2} \cdot A_2 n = \frac{m_2}{m_1 + m_2}(z_2 - z_1).$$

But the distance of g_1 from L is equal to

$$z_1 + g_1 m = z_1 + \frac{m_2}{m_1 + m_2}(z_2 - z_1) = \frac{m_1 z_1 + m_2 z_2}{m_1 + m_2}.$$

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* All this holds if the points A_1, A_2, ... are not in the same plane and L represents any plane from which their distances are measured.

Calling this distance $\bar{z}_1$, we have the distance of g_2 from L equal to

$$\frac{(m_1+m_2)\bar{z}_1+m_3z_3}{m_1+m_2+m_3}=\frac{m_1z_1+m_2z_2+m_3z_3}{m_1+m_2+m_3},$$

since g_1A_3 is divided at g_2 in the ratio $\frac{m_3}{m_1+m_2}$. Continuing the application of this method, we have evidently, if $\bar{z}$ is the distance of G from L,

$$\bar{z}=\frac{m_1z_1+m_2z_2+m_3z_3+\ldots m_nz_n}{m_1+m_2+m_3+\ldots+m_n}, \tag{1}$$

or

$$\bar{z}=\frac{\Sigma mz}{\Sigma m}, \tag{2}$$

where Σ is the sign of summation.

The point G thus arrived at is called *The Centre of Mean Position of the given points for the system of multiples* m_1, m_2, m_3, $\ldots m_n$.

The points A_1, A_2, A_3, ... remaining the same, and the system of multiples being altered to p_1, p_2, p_3, ... the point G arrived at would, of course, be different. The distance of the new point would be $\frac{\Sigma pz}{\Sigma p}$.

In particular, the distance, $\bar{z}$, from any plane, of the centre of parallel forces, P_1, P_2, P_3, ... applied, respectively, at points whose distances from the plane are z_1, z_2, z_3, ... is given by the equation

$$\bar{z}=\frac{P_1 . z_1+P_2 . z_2+P_3 . z_3+\ldots}{P_1+P_2+P_3+\ldots}$$

or

$$\bar{z}=\frac{\Sigma Pz}{\Sigma P}.$$

The construction given in this Article for the Centre of Mean Position of the points A_1, A_2, A_3, ... holds evidently when the points do not all lie in one plane, as previously remarked, so that if z_1, z_2, z_3, ... denote the distances of the points from an arbitrary *plane*, the distance, $\bar{z}$, of the centre of mean position from this plane, for the system of multiples m_1, m_2, m_3, ..., is given by the equation

$$\bar{z}=\frac{\Sigma mz}{\Sigma m}.$$

Centre of Mean Position is a generic term which comprises under it particular points which must be specially noticed. One, *the Centre of Parallel Forces*, has been already mentioned. Another is the *Centre of Mass*, called also the *Centre of Inertia.*

If at the points considered, A_1, A_2, A_3, ... there be placed material particles whose masses are respectively m_1, m_2, m_3, ... and we find the centre of mean position of these points for the system of multiples m_1, m_2, m_3, ... we shall arrive at the *Centre of Mass* of this system of particles. Nothing is here assumed about the closeness of the points A_1, A_2, A_3, ..., or the particles placed at them, and the process of arriving at the point G will be unaltered if these particles constitute a continuous body. Hence the *Centre of Mass* of any body is *the Centre of Mean Position of all the points within it for a system of multiples proportional to the masses of the infinitely small particles placed at these points respectively.*

A body whose points do not suffer any relative changes of position will therefore continue to possess the same centre of mass no matter into what part of the universe the body may be taken. A different arrangement of its particles, would, of course, in general alter its centre of mass. The centre of mass of a rigid body is, then, something which it possesses absolutely, or apart from all contingency of position in space or relation to other bodies.

The distance of this point from any plane is given by the equation last written, in which the sign Σ is to be replaced by the integral sign $\int$, and the element of mass at a distance z from the plane denoted by dm. Thus

$$\bar{z} = \frac{\int z dm}{\int dm}.$$

Again, if at the points A_1, A_2, A_3,... there be placed particles whose *weights* are w_1, w_2, w_3, ... these weights constituting a system of parallel forces, the centre of these parallel forces is called the *Centre of Gravity* of the given particles.

The effect of altering the position of the body in the most general manner possible is merely to turn the forces, w_1, w_2, w_3,... round their fixed points of application A_1, A_2, ... through the same angle, and by the last article we see that the resultant of the weights of the particles will, in all positions of the body, pass through a fixed point, G, in the body. The resultant of all the elementary weights is equal to their sum, and is called the *weight of the body*. We may, therefore, define the centre of gravity of a body thus—*The centre of gravity of a body is that unique point in it through which passes, in all possible positions*

4. If L, M, N are the sums of the moments of a system of coplanar forces about three points A, B, C, show that the equation of the line of action of the resultant in trilinear co-ordinates referred to the triangle ABC is

$$aLx + bMy + cNz = 0,$$

where a, b, c are the lengths of the sides of the triangle.

5. Find the magnitude of the resultant in terms of L, M, N.

Ans. $$R^2 = \frac{1}{4\Delta^2}\Sigma a^2(L-M)(L-N),$$

where Δ is the area of ABC, whose sides are a, b, c.

86.] **Conditions of Equilibrium of a Rigid Body acted upon by Coplanar Forces.** When a rigid body is acted upon by a system of coplanar forces, this system is reducible, generally, to a single force, but it may happen to reduce to a couple. The necessary and sufficient condition of equilibrium of the body evidently is that *this resultant of the forces shall be of zero magnitude without acting at infinity;* and the latter part of this condition will be guaranteed by the vanishing of the sum of the moments of the forces round any one point in their plane. Indeed, if the forces have no moment round any three points which do not lie in one right line, they cannot reduce either to a single force or to a couple; so that we may say that—*the necessary and sufficient conditions of equilibrium of a system of coplanar forces acting on a rigid body are that they give zero moment round each of any three points which do not lie in one right line.* (α)

Their equilibrium will also be guaranteed—i. e., they must have a zero resultant without reducing to a couple—if *the sums of their components, rectangular or oblique, vanish along any two particular lines in their plane, and also the sum of their moments about any one point vanishes.* (β)

For, if they have no total component along any direction (L) and also no total component along any other direction (L'), their resultant must be of zero magnitude, since its component along any direction is equal to the sum of the components of the forces in that direction; and if, in addition, their total moment round any one point vanishes, they cannot reduce to a couple.

When, therefore, a system of coplanar forces acting on a rigid body is in equilibrium we have the result that—

$$\left.\begin{array}{l}(1)\ \textit{the sum of their components along every line} = 0;\\ (2)\ \textit{the sum of their moments about every point} = 0.\end{array}\right\}\quad(\gamma)$$

The conditions which have just been enunciated are both necessary and *sufficient* for the equilibrium of any *rigid* material system to which the forces may be applied; but they are not sufficient for the equilibrium of a material system whose parts are not rigidly connected. The simplest way of seeing this is to remember that the whole process of compounding forces, two by two, into a single resultant is invalid and, indeed, without any meaning, unless they are all applied to one and the same rigid body. This was pointed out at the very outset (see p. 17). If a force, P, is applied at a point to a bar which is freely jointed to another bar to which a force P_2 is applied, the process of finding the resultant of P_1 and P_2 consists in producing their lines of action to meet in a point O, and then assuming that both P_1 and P_2 can be applied at O. But this assumes that O is at once rigidly connected with both bars, i. e., that the two bars are rigidly connected with each other; otherwise we cannot speak at all of the *resultant* of P_1 and P_2.

Let us imagine, then, any non-rigid, or deformable, system—a deformable framework of freely jointed bars, an elastic string, a fluid, &c.—at rest, and suppose that we are about to apply to this system a system of forces, $P_1, P_2, P_3, \ldots$ at specified points: what conditions must these forces satisfy in order that, when they are applied, the material system will continue to be at rest? Now, although we cannot completely answer the question without regard to the special features and structure of the material system, this much we *can* say, that—if we imagine the whole system to be suddenly made rigid, and if the forces will not keep this rigidified system at rest, they will, *à fortiori*, not suffice to keep the given deformable system at rest; for, rigidification always removes restrictions which hamper the action of the internal forces of the system, and renders these forces, therefore, more capable of preserving the equilibrium of the various connected parts against the action of external forces.

Hence the conditions expressed in (α), (β), (γ), though *necessary* for the equilibrium of forces applied to a deformable system, are not *sufficient*.

Analytical expression may be given to any of the forms (α), (β), (γ) which express the conditions of equilibrium. Thus, to express (β), let O (Fig. 98) be the point about which we take moments, and through O draw any two rectangular lines, Ox, Oy, which we may employ as axes of co-ordinates for the points of application of the various forces. Let P_1 act at A_1 and resolve the force P_1 into two components, X_1 and Y_1, parallel to Ox and Oy. Now (Art. 82), the moment of P_1 about O is equal to the sum of the moments of X_1 and Y_1 about O. But, if rotation opposite to that of a watch-hand is considered positive, the moment of Y_1 about O is $Y_1 . x_1$; and the moment of X_1 is $-X_1 . y_1$, where x_1 and y_1 are the co-ordinates of A_1 referred to the axes Ox and Oy. Hence the moment of P_1 about O is

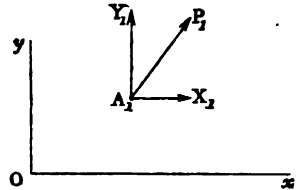

Fig 98.

$$Y_1x_1 - X_1y_1.$$

Adding together the moments of P_1, P_2, ..., we get the total moment

$$G = \Sigma(Yx - Xy). \qquad (1)$$

If the sum of the components of the forces along Ox is denoted by ΣX, and the sum of the components along Oy by ΣY, the resultant of the forces (Art. 84) is given by the equation

$$R^2 = (\Sigma X)^2 + (\Sigma Y)^2. \qquad (2)$$

Now, since for equilibrium we must have $R = 0$, and $G = 0$, the conditions, analytically expressed, are

$$\Sigma X = 0, \Sigma Y = 0, \qquad (3)$$
$$\Sigma(Yx - Xy) = 0. \qquad (4)$$

When the student comes to the study of the kinetics of a material system, he will see that if the external forces applied to it are such that $R = 0$ and $G = 0$, i. e., are such that if they acted on a *rigid* system, their resultant force and total moment round any axis both vanish, the centre of mass of the system is at rest (or in uniform motion in a fixed right line) and there is no resultant moment of mass-acceleration round any axis. This is true for all kinds of material systems, whether they are gases, liquids, deformable frameworks, natural solids, or rigid bodies.

Now the destruction of resultant linear and angular acceleration will, except in the case of rigid bodies, be quite consistent with the existence of motions of parts of the system among themselves, negative momenta cancelling positive. Hence, *whenever a system is capable of altering the relative positions of its parts, the complete equilibrium* *of the system will require more than the vanishing of the resultants of translation and rotation of the forces applied to it.* In fact, its internal forces will have to be taken into account. In rigid bodies the destruction of the above-mentioned motions will necessitate the destruction of all motion, and the conditions $R = 0$, $G = 0$ are both necessary and sufficient. In these bodies there is no restriction placed on the internal forces, so that they are always capable of assuming such magnitudes and directions as will enable them to destroy the action of the external forces. On the contrary, in deformable bodies, there are restrictions placed on the internal forces so that they are not capable of preserving equilibrium against all systems of external forces. For example, in a freely jointed framework, the action between bar and bar must consist of a single force restricted to passing through the joint. This is the reason why two equal forces applied in opposite senses in the same line to two opposite sides of a set of parallel rulers will not hold them in equilibrium, unless the rulers are placed in a certain configuration; and it is also the reason why two equal and directly opposed forces applied to the ends of a string, elastic or inelastic, will not hold it in equilibrium, until it has assumed a certain state.

Hence also the necessity for considering the internal forces (pressures) in Hydrostatics.

Nevertheless, the conditions of equilibrium of *all* material systems whatever—natural solids, liquids, and gases—are completely expressed by the single principle that—*when the system has assumed its configuration of equilibrium, then for all imaginable small derangements of its parts the whole work which would thereby be done by all the acting forces, external and internal, is zero*—which is Lagrange's great principle of Virtual Work.

87. **Reduction of a system of Forces.** We have seen that a system of coplanar forces must ultimately reduce either to a single force or to a couple. It can be reduced to a single force acting at *any* point, O, supposed rigidly connected with the body, accompanied by a couple. For, let the system reduce to a single force, R, acting in a line (L). At O introduce two forces $(R, -R)$ parallel and equal to R and acting in opposite senses. Take R acting in the line (L) and $-R$ acting at O to make a couple; then, in addition to this couple, we have the force R at O. If p is the perpendicular from A on (L), the moment of the couple is

$$R . p,$$

i. e., it is the sum of the moments of the given forces about O. Thus the force at O is constant in magnitude and direction whatever point, O, is chosen, while the moment of the couple is variable.

The same thing is easily seen, less directly, by replacing each separate force $P_1, \ldots$ by a force $P_1, \ldots$ acting at O, accompanied by a couple (see Theorem VII. p. 111).

Thus, take the special case in which all the forces are parallel (Fig. 99).

Take any point O, and draw through it a right line, Oy, parallel to the forces. At O introduce two forces, P_1' and P_1'', each equal to P_1, these new forces being directly opposed to each other along Oy. Now, P_1 and P_1'' form a couple whose moment is $P_1 . p_1$, if p_1 is the perpendicular from O on the line A_1P_1. Introducing in the same way, two forces, P_2' and P_2'', equal to P_2, directly opposite to each other along Oy, we have P_2 at A_2 replaced by a force P_2'' acting at O along Oy' and a couple whose moment is $-P_2 . p_2$, p_2 being the perpendicular from O on the line A_2P_2. The — sign is attached to this couple because the couple (P_2', P_2) tends to produce rotation in a sense opposite to that in which the couple (P_1'', P_1) tends to produce rotation.

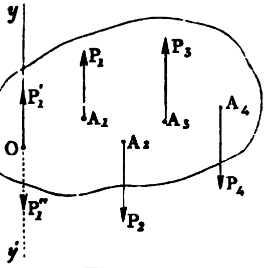

Fig. 99.

Proceeding in this way with all the forces in the above figure, we have the whole system of forces at A_1, A_2, A_3, $A_4, \ldots$ equivalent to a single force,

$$P_1 - P_2 + P_3 - P_4 + \ldots,$$

acting at O in the direction Oy, and a couple,

$$P_1 . p_1 - P_2 . p_2 + P_3 . p_3 - P_4 . p_4 + \ldots,$$

tending to turn the body round O in a sense opposite to that of watch-hand rotation.

Denoting the resultant force by R, and the moment of the resultant couple by G, we have

$$R = \Sigma P, \tag{1}$$

$$G = \Sigma (P . p). \tag{2}$$

Again, let the forces act in any directions.

At O (Fig. 100) introduce two opposite forces, P_1' and P_1'', each equal and parallel to P_1. Let P_1 and P_1'' be considered as forming a couple. Then P_1 at A_1 is equivalent to P_1 acting at O, and a couple whose moment $= P_1 . p_1$. Replace P_2 at A_2 in the same way by P_2'' (or P_2) acting at O, and a couple (P_2, P_2') whose moment is $- P_2 . p_2$. Thus the whole system of forces will be replaced by forces P_1, P_2, P_3, $P_4, \ldots$, acting at O, and a number of couples

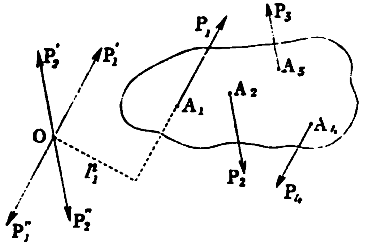

Fig. 100.

whose moments are $P_1 . p_1$, $-P_2 . p_2$, $P_3 . p_3$, $-P_4 . p_4$,... (the forces acting as in the above figure). The forces acting at O will have a single resultant, R, and the couples will form a single couple whose moment, G, is (Theor. VIII, Art. 80) the sum of the moments of the couples.

88.] **Equation of line of action of Resultant.** The equation of the line of action of the resultant of a system of coplanar forces is obtained with reference to any assumed axes of coordinates, Cartesian or Trilinear, by expressing the fact that the sum of the moments of the given forces round any point on the line of action of R is zero.

Thus, let Ox, Oy (Fig. 98) be any rectangular axes with respect to which the co-ordinates of the points of application of P_1, P_2, ... are (x_1, y_1), (x_2, y_2), ..., and let (x, y) be the coordinates of any point in the plane. Then the moment of P_1 about (x, y) is evidently

$$Y_1(x_1-x)-X_1(y_1-y),$$

or

$$Y_1x_1-X_1y_1-xY_1+yX_1.$$

The moment of P_2 about (x, y) is

$$Y_2x_2-X_2y_2-xY_2+yX_2,$$

and similarly for the other forces. Hence the total moment, G', of the forces about the point is given by the equation

$$G' = G-x\Sigma Y+y\Sigma X, \tag{1}$$

where G is the sum of the moments of the forces about O and stands for $(Y_1x_1-X_1y_1) + (Y_2x_2-X_2y_2) + \dots$.

If (x, y) is any point on R, $G' = 0$, so that the equation of the line of action of R is

$$x\Sigma Y-y\Sigma X = G. \tag{2}$$

The interpretation of equation (1) is obviously that—*The sum of the moments of a system of coplanar forces about any point, O, is equal to the sum of their moments about any other point, O′, plus the moment about O of their resultant of translation, supposed acting at O′*—a result which is evident from first principles.

EXAMPLES.

1. O is any point inside a triangle ABC; α, β, γ are the perpendiculars from O on the sides; if forces act along the sides a, b, c equal to $\frac{k}{\alpha}$, $\frac{k}{\beta}$, $\frac{k}{\gamma}$, respectively, and all in the same cyclical order, find the line of their resultant.

Take any point whose trilinear co-ordinates with reference to the triangle ABC are x, y, z; then the sum of the moments of the forces about it is $\frac{k}{\alpha}x + \frac{k}{\beta}y + \frac{k}{\gamma}z$. Hence the equation of the line of R is

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 0,$$

which is the axis of homology of ABC and the triangle $A'B'C'$ formed by the points in which AO, BO, CO intersect the opposite sides.

If two of the forces, $\frac{k}{\alpha}$, $\frac{k}{\beta}$, have the same cyclical order while the third acts in the reverse order, the equation of R is

$$\frac{x}{\alpha} + \frac{y}{\beta} - \frac{z}{\gamma} = 0,$$

i.e., R acts in the line $A'B'$.

2. A triangular plate, ABC (Fig. 101), is acted upon at each angle by forces, along the two sides containing it, represented in magnitudes and lines of action by the distances between the angle and the feet of the perpendiculars let fall from the other two angles on these sides. Find the line of action of the resultant force.

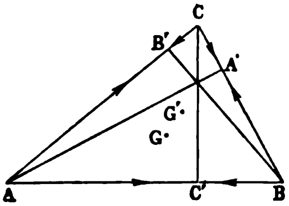

Fig. 101.

Let the perpendiculars let fall on the three sides, a, b, c, from any point, P, on the resultant be x, y, z, respectively, and let A', B', C' be the feet of the perpendiculars. Then the force in AB in the sense AB is $AC'-BC'$, or $b\cos A - a\cos B$. Hence the moment of this force about P is $z(b\cos A - a\cos B)$, and since the sum of the moments of all the forces (estimated in cyclical order) round P is $= 0$ (Art. 82), we have

$$x(c\cos B - b\cos C) + y(a\cos C - c\cos A) + z(b\cos A - a\cos B) = 0. \tag{1}$$

Now, one set of values of x, y, and z, which will satisfy this equation, is, evidently, a, b, c. Hence the resultant passes through the point the perpendiculars from which on the sides are proportional to a, b, c. This point is thus found:—Let G be the centre of gravity of the triangle; from A draw a line, AG', which makes $\angle CAG' = \angle BAG$, and from B draw a line, BG', which makes $\angle CBG' = \angle ABG$. These lines intersect in G', the required point which is called the isogonal conjugate of G, or the *Symmedian point.*

Again, another set of values of x, y, z, which will satisfy (1), is $\cos A$, $\cos B$, $\cos C$; and the resultant passes through the point whose perpendiculars on the sides are proportional to these quantities. This point is the centre of the circumscribed circle.

Hence the line of action of the resultant is known.

3. Show that the resultant of the system of forces in the last example is

$$\frac{4\Delta}{abc}\sqrt{a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2},$$

where Δ is the area of the triangle.

4. Forces P, Q, R, act along the sides of a triangle, ABC, and their resultant passes through the centres of the inscribed and circumscribed circles: prove that

$$\frac{P}{\cos B-\cos C}=\frac{Q}{\cos C-\cos A}=\frac{R}{\cos A-\cos B}$$

(Wolstenholme's Book of Mathematical Problems).

5. If from a point C on a circle there be drawn two chords CA, CB, prove that two forces inversely proportional to AC and BC acting along these lines both to, or both from, C, have a resultant passing through the Symmedian point of ABC and inversely proportional to the chord through this point.

6. If AD and BC (A, B, C, D being in cyclical order) are any two chords of a circle and two forces inversely proportional to them act along them in the senses AD, BC, their resultant passes through the intersection of AC and BD; but if they act in the senses AD, CB, it passes through the intersection of AB and DC.

7. If forces inversely proportional to the trilinear co-ordinates of any point on the circle circumscribing a triangle act, in cyclical order, along the corresponding sides, prove that their resultant will always pass through the isogonal conjugate of the centroid of the triangle.

8. Find the relation between three forces such that if they act along the sides of a triangle in cyclical order, their resultant will always touch the circumscribed circle.

Ans. If the forces are P, Q, R, and a, b, c are the sides,

$$\sqrt{aP}+\sqrt{bQ}+\sqrt{cR}=0,$$

so that the trilinear co-ordinates of any point on the inscribed conic which touches the sides at their middle points will represent all such force systems.

9. Find the relation between the forces if their resultant always touches the inscribed circle.

$$\textit{Ans.}\qquad \frac{\cos^2\frac{1}{2}A}{P}+\frac{\cos^2\frac{1}{2}B}{Q}+\frac{\cos^2\frac{1}{2}C}{R}=0,$$

so that they can be represented by the trilinear co-ordinates of any point on a certain conic circumscribing the triangle.

(These results can be obtained purely by the principle of moments, Art. 82.)

10. Generally, if the resultant of forces acting in cyclical order

along the sides of a triangle touches any assigned curve, the relation between the forces is expressed by the tangential equation of the curve in trilinear co-ordinates.

89.] **Centre of Parallel Forces. Theorem.** *If any number of parallel forces,* $P_1, P_2, P_3, \dots P_n$, *act in one plane at points* $A_1, A_2, A_3, \dots A_n$, *their resultant passes through a fixed point if all the forces are turned in the same sense round their points of application through an arbitrary but common angle.*

The point, g_1 (Fig. 102), of application of the resultant of P_1 and P_2 has been determined (Art. 75) by dividing the line $A_1 A_2$ so that

$$\frac{A_1 g_1}{A_2 g_1} = \frac{P_2}{P_1},$$

on the supposition that the forces P_1 and P_2 are parallel, but no assumption has been made as to their common direction. Hence g_1 will be a point on their resultant in whatever direction they act, and the force at this point is $P_1 + P_2$. The point of application of the resultant of P_1, P_2, and P_3, is determined by joining g_1 to A_3, and dividing it in g_2, so that

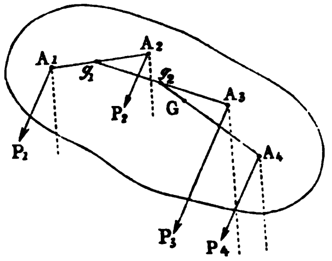

Fig. 102.

$$\frac{g_1 g_2}{A_3 g_2} = \frac{\text{force at } A_3}{\text{force at } g_1} = \frac{P_3}{P_1 + P_2},$$

and the force at g_2 is $P_1 + P_2 + P_3$. Similarly, the point of application of the resultant of P_1, P_2, P_3, and P_4, is a point, G, on $g_2 A_4$, such that

$$\frac{g_2 G}{A_4 G} = \frac{P_4}{P_1 + P_2 + P_3},$$

and the force at $G = P_1 + P_2 + P_3 + P_4$.

We thus see that the point, G, of application of the resultant of the system is determined by dividing the lines $g_1 A_3$, $g_2 A_4$, ... in certain ratios which depend simply on the *magnitudes*, and not on the common *direction*, of the forces at A_1, A_2, A_3, ... The theorem is, therefore, evident.

Of course no one point on the line of action of a force which acts on an indeformable body has a special right to be called *the* point of application of the force; and if we are given a parallel system, P_1, P_2, P_3, ... with a fixed direction, *a* point on their resultant can be found by taking *any* points, B_1, B_2, B_3, ... on their respective lines of action and going through the above process; but the point so determined on the resultant is not one through which this resultant would continue to pass if the common direction of the parallel system were changed in such a way that the forces, respectively, revolved round a different system of points, A_1, A_2, A_3, ... If we contemplate such a revolution round these points, the point G is appropriately termed *the* centre of the system.

90.] **Centre of Mean Position.** Let there be any number of points, A_1, A_2, A_3, ... (Fig. 103), in one plane, and let the line, A_1A_2, be divided at g_1 so that

$$\frac{g_1A_2}{g_1A_1} = \frac{m_1}{m_2};$$

let g_1A_3 be divided at g_2, so that

$$\frac{g_2A_3}{g_2g_1} = \frac{m_1+m_2}{m_3};$$

let g_2A_4 be divided at g_3, so that

$$\frac{g_3A_4}{g_3g_2} = \frac{m_1+m_2+m_3}{m_4};$$

and so on, until by a final construction we arrive at a point G. It is required to express the distance of G from an arbitrary line, L, in the plane of the points in terms of the distances, $z_1, z_2, z_3, \ldots$ of A_1, A_2, A_3, ... from this line.*

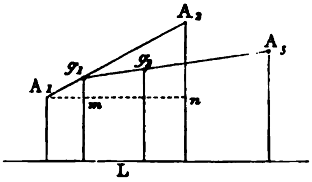

Fig. 103.

Draw A_1mn parallel to L. Then

$$\frac{g_1m}{A_2n} = \frac{A_1g_1}{A_1A_2} = \frac{m_2}{m_1+m_2},$$

$$\therefore \quad g_1m = \frac{m_2}{m_1+m_2}\cdot A_2n = \frac{m_2}{m_1+m_2}(z_2-z_1).$$

But the distance of g_1 from L is equal to

$$z_1+g_1m = z_1+\frac{m_2}{m_1+m_2}(z_2-z_1) = \frac{m_1z_1+m_2z_2}{m_1+m_2}.$$

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* All this holds if the points A_1, A_2, ... are not in the same plane and L represents any plane from which their distances are measured.

Calling this distance $\bar{z}_1$, we have the distance of g_2 from L equal to

$$\frac{(m_1+m_2)\bar{z}_1+m_3z_3}{m_1+m_2+m_3}=\frac{m_1z_1+m_2z_2+m_3z_3}{m_1+m_2+m_3},$$

since g_1A_3 is divided at g_2 in the ratio $\frac{m_3}{m_1+m_2}\cdot$ Continuing the application of this method, we have evidently, if $\bar{z}$ is the distance of G from L,

$$\bar{z}=\frac{m_1z_1+m_2z_2+m_3z_3+\ldots m_nz_n}{m_1+m_2+m_3+\ldots+m_n}, \qquad (1)$$

or

$$\bar{z}=\frac{\Sigma mz}{\Sigma m}, \qquad (2)$$

where Σ is the sign of summation.

The point G thus arrived at is called *The Centre of Mean Position of the given points for the system of multiples* $m_1, m_2, m_3, \ldots m_n$.

The points $A_1, A_2, A_3, \ldots$ remaining the same, and the system of multiples being altered to $p_1, p_2, p_3, \ldots$ the point G arrived at would, of course, be different. The distance of the new point would be $\frac{\Sigma pz}{\Sigma p}$.

In particular, the distance, $\bar{z}$, from any plane, of the centre of parallel forces, P_1, P_2, P_3, ... applied, respectively, at points whose distances from the plane are z_1, z_2, z_3, ... is given by the equation

$$\bar{z}=\frac{P_1\cdot z_1+P_2\cdot z_2+P_3\cdot z_3+\ldots}{P_1+P_2+P_3+\ldots}$$

or

$$\bar{z}=\frac{\Sigma Pz}{\Sigma P}.$$

The construction given in this Article for the Centre of Mean Position of the points A_1, A_2, A_3, ... holds evidently when the points do not all lie in one plane, as previously remarked, so that if z_1, z_2, z_3, ... denote the distances of the points from an arbitrary *plane*, the distance, $\bar{z}$, of the centre of mean position from this plane, for the system of multiples m_1, m_2, m_3, ..., is given by the equation

$$\bar{z}=\frac{\Sigma mz}{\Sigma m}.$$

Centre of Mean Position is a generic term which comprises under it particular points which must be specially noticed. One, *the Centre of Parallel Forces*, has been already mentioned. Another is the *Centre of Mass*, called also the *Centre of Inertia.*

If at the points considered, A_1, A_2, A_3, ... there be placed material particles whose masses are respectively m_1, m_2, m_3, ... and we find the centre of mean position of these points for the system of multiples m_1, m_2, m_3, ... we shall arrive at the *Centre of Mass* of this system of particles. Nothing is here assumed about the closeness of the points A_1, A_2, A_3,..., or the particles placed at them, and the process of arriving at the point G will be unaltered if these particles constitute a continuous body. Hence the *Centre of Mass* of any body is *the Centre of Mean Position of all the points within it for a system of multiples proportional to the masses of the infinitely small particles placed at these points respectively.*

A body whose points do not suffer any relative changes of position will therefore continue to possess the same centre of mass no matter into what part of the universe the body may be taken. A different arrangement of its particles, would, of course, in general alter its centre of mass. The centre of mass of a rigid body is, then, something which it possesses absolutely, or apart from all contingency of position in space or relation to other bodies.

The distance of this point from any plane is given by the equation last written, in which the sign Σ is to be replaced by the integral sign $\int$, and the element of mass at a distance z from the plane denoted by dm. Thus

$$\bar{z} = \frac{\int z dm}{\int dm}.$$

Again, if at the points A_1, A_2, A_3,... there be placed particles whose *weights* are w_1, w_2, w_3, ... these weights constituting a system of parallel forces, the centre of these parallel forces is called the *Centre of Gravity* of the given particles.

The effect of altering the position of the body in the most general manner possible is merely to turn the forces, w_1, w_2, w_3,... round their fixed points of application A_1, A_2, ... through the same angle, and by the last article we see that the resultant of the weights of the particles will, in all positions of the body, pass through a fixed point, G, in the body. The resultant of all the elementary weights is equal to their sum, and is called the *weight of the body*. We may, therefore, define the centre of gravity of a body thus—*The centre of gravity of a body is that unique point in it through which passes, in all possible positions* *of the body, the resultant of the system of parallel forces formed by the weights of the indefinitely great number of indefinitely small particles into which the body can be divided.*

The centre of gravity of a body is, then, the centre of the particular set of parallel forces which act on its various elements in virtue of the attraction of the Earth. The existence of such a point depends on the parallelism of the forces produced by the Earth on the elements of the body, and this parallelism, again, depends on the minuteness of the volume of the body in comparison with that of the Earth. If the body were carried to the surface of the Sun, or any other such large attracting mass, the individual weights of its infinitesimal elements, and therefore its total weight, would be greater than they are at the Earth's surface, but the position of the centre of gravity in the body would remain the same. On the other hand, if the dimensions of the body were comparable with those of the attracting mass, the forces of attraction on its elementary portions would not be a parallel system, and the resultant attraction would not, in general, pass through any fixed point in the body independently of the relative positions of the two masses. This resultant attraction on the body would not, except for very special positions of the body, pass through its *centre of mass*. The term *weight of a body* is used to signify the resultant attraction produced on the body by the Earth, or other planet, on whose surface the body exists, and it is therefore, unlike mass, a mere contingent property of the body. If we imagine the body taken out into space and removed (if possible) from the attractions of all bodies, the terms *weight* and *centre of gravity* would cease to have any meaning with reference to the body in that position; while, on the contrary, it has both its *mass* and *centre of mass* perfectly unaltered. Hence the centre of gravity is *essentially* distinguished from the centre of mass; although, since the force with which any body, A, attracts each molecule of another body, B, is proportional to the mass of the molecule, if these forces form a parallel system—as they do in the case of a planet and any comparatively small body on its surface—the centre of this gravitation system coincides with the centre of mass of the attracted body.

In considering the equilibrium of a rigid heavy body we represent its weight as a single force acting vertically through its centre of gravity.

For the actual process of calculating the distance of the centre of mass of a given system of masses, or the centre of a given system of parallel forces, from any plane of reference, the student will often find it useful to make a table of the masses or the forces and the distances of their corresponding points from the plane, as in the following scheme:

Masses.	Distances from plane.	Products.
m_1	z_1	$m_1.z_1$
m_2	z_2	$m_2.z_2$
m_3	z_3	$m_3.z_3$
...	...	...
...	...	...
M		P

In the first column are placed successively the constituent masses of the system; opposite each mass is placed, in the second column, the distance of its centre of mass from the reference plane; and in the third column are placed the products of the corresponding elements of the first and second columns. If the sum of the third column is P and that of the first M,

$$\frac{P}{M}$$

is the distance of the centre of mass of the system from the reference plane. We shall subsequently refer to this result as the *theorem of mass-moments* with respect to a plane.

If instead of a system of masses we have a system of parallel forces, the heading of the first column will be *Forces*.

If some of the centres of mass of the bodies m_1, m_2, ... are at one side of the plane and others at the other side, the corresponding distances in the second column must be marked positive and negative. In the case of a system of parallel forces if those in the same sense are taken as positive, those in the opposite sense must be marked negative in the first column.

A single typical example will suffice.

ABC is a triangle whose sides *a*, *b*, *c* are, respectively, 13, 15, 14 inches long. At the vertices *A*, *B*, *C* act like parallel forces of 12, 4, and 10 pounds' weight, respectively; and at the middle points of the opposite sides act forces of 6, 8, and 2 pounds' weight, parallel to the first system and all having a sense opposite to that of the first system.

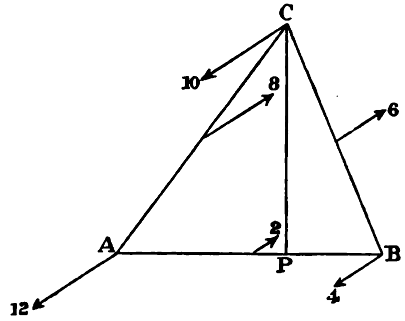


Fig. 104.

It is required to find the distance of the *centre* of this whole system from the perpendicular, *CP*, drawn from *C* to *AB*.

We find $CP = 12$, $AP = 9$, $BP = 5$, and, setting down in a table the magnitudes of the forces and the distances of their points of application from *CP* (i.e., in reality, from a plane through *CP* perpendicular to the plane of the figure) we have :

Forces.	Distances from *CP*.	Products.
12	9	108
-8	$\frac{9}{2}$	-36
10	0	0
-6	$-\frac{5}{2}$	15
4	-5	-20
-2	2	-4
10		63

so that 6.3 is the distance of the centre from *CP*, distances from *CP* towards the left of the figure having been taken as positive.

If we wish to find the distance of the centre from *AB*, we join a column of distances from *AB* to the above table and a corresponding column of products, the sum of which products we find to be 36, so that the centre is 3.6 from *AB*.

EXAMPLES.

1. The centre of mean position of three points, *A*, *B*, *C*, for a system of equal multiples, is the intersection of the bisectors of the sides of the triangle *ABC* drawn from the opposite angles.

2. The centre of mean position of three points, A, B, C, for a system of multiples $\sin 2A$, $\sin 2B$, $\sin 2C$, is the centre of the circle circumscribed about the triangle ABC.

3. The sides of the triangle being a, b, c, the centre of mean position of A, B, C, for the system of multiples a, b, c, is the centre of the inscribed circle, and if any one of these multiples is taken negatively, the mean centre is the centre of the circle exscribed at the corresponding side.

4. For the system of multiples $\tan A$, $\tan B$, $\tan C$, the centre of mean position is the intersection of perpendiculars.

5. For the system of multiples a^2, b^2, c^2 the mean centre of A, B, C is the isogonal conjugate of G, i.e., the Symmedian point.

6. O is any point inside or outside a triangle ABC. Show that if a system of multiples of any kind proportional to the areas BOC, COA, and AOB be placed at the vertices A, B, C, respectively, the point O is the mean centre of the system. (If O is outside, one of the areas above must be considered as negative.)

7. O is any point, and P, Q, R are the feet of the perpendiculars from it on the sides a, b, c of a triangle ABC. Show that if a system of multiples proportional to the areas QOR, ROP, POQ be placed at A, B, C, respectively, the mean centre is the isogonal conjugate of O with respect to the triangle. (If O' is this point, its definition is that $\angle BAO' = \angle CAO$, $\angle CBO' = \angle ABO$, &c.)

8. Hence show that the feet of the perpendiculars from any point on the circumscribed circle lie in a right line.

(The isogonal conjugate of any point on this circle is at infinity.)

9. O is any point inside a triangle ABC, and the lines AO, BO, CO meet the opposite sides in A', B', C', respectively. Show that it is possible to determine masses m_1, m_2, m_3 at A, B, C, so that if they are, respectively, moved to P, Q, R along AA', BB', CC' such that $\dfrac{AP}{AA'} = \dfrac{BQ}{BB'} = \dfrac{CR}{CC'}$ their centre of mass remains unchanged in position, and that

$$m_1 : m_2 : m_3 = \frac{OA \,.\, OA'}{AA'^2} : \frac{OB \,.\, OB'}{BB'^2} : \frac{OC \,.\, OC'}{CC'^2}.$$

10. Show that $m_1 : m_2 : m_3$ will have the following values if—

O is the centroid of ABC . . . $1 : 1 : 1$,

„ „ isogonal conjugate of centroid of ABC

$$a^2(b^2+c^2) : b^2(c^2+a^2) : c^2(a^2+b^2),$$

„ „ orthocentre . . . . . $a^2 : b^2 : c^2$,

„ „ centre of inscribed circle . $a(b+c) : b(c+a) : c(a+b)$,

„ „ isogonal conjugate of any point, P,

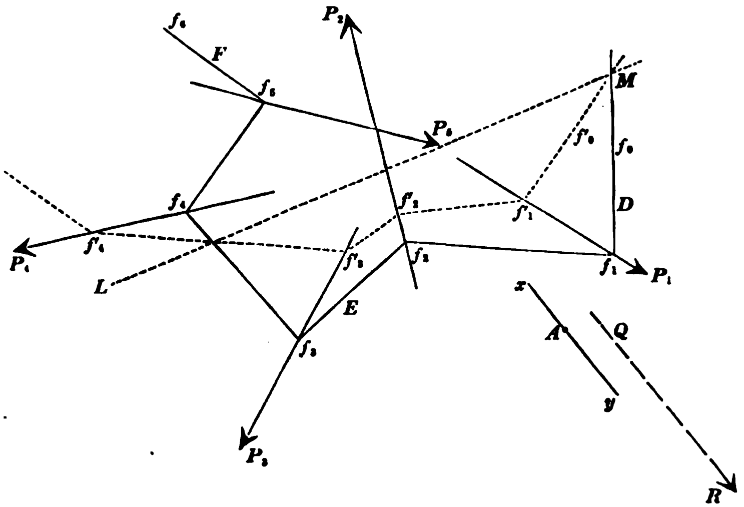
$$\frac{a}{\alpha}\left(\frac{b}{\beta}+\frac{c}{\gamma}\right) : \frac{b}{\beta}\left(\frac{c}{\gamma}+\frac{a}{\alpha}\right) : \frac{c}{\gamma}\left(\frac{a}{\alpha}+\frac{b}{\beta}\right),$$

where α, β, γ are the perpendiculars from P on the sides.

11. If at given points, A_1, A_2, A_3, ... there be placed a system of multiples, m_1, m_2, m_3, ... whose mean centre is P; and if when multiples μ_1, μ_2, μ_3, ... are placed at the same points the mean centre is Q; prove that for a system of multiples $m_1+k\mu_1$, $m_2+k\mu_2$, $m_3+k\mu_3$, ... at the points the mean centre lies on the line PQ, dividing it in the ratio $\Sigma m : k\Sigma\mu$.

Hence, with the aid of examples 5 and 6, prove that the centroid of a triangle lies on the line joining the orthocentre to the centre of the circumscribed circle, cutting it in a point of trisection. (Observe that the orthocentre and centre of the circumscribed circle are isogonal conjugates.)

91.] **Force Polygon and Funicular Polygon.** The previous methods of this chapter for the determination of the magnitude and line of action of the resultant of a system of coplanar forces have been analytical. We now proceed to show how the resultant can be determined by a graphic method which is constantly employed in calculations relating to various engineering structures. Let there be any system of forces, P_1, P_2, P_3, P_4, P_5, (Fig. 105) acting in one plane on a body. Starting



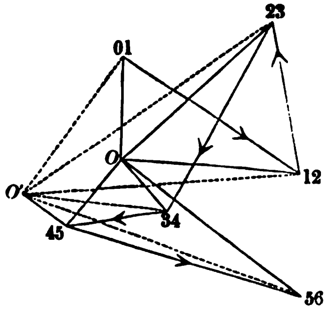


Fig. 105.

with any point, 01, draw lines, (01, 12), (12, 23), (23, 34), (34, 45), (45, 56), parallel to the lines of action of the forces and respectively proportional to them. The figure formed by these lines, (01, 12), (12, 23), ..., is called the *Force Polygon* of the given system of forces. Now take any point, O, and from it draw lines, O 01, O 12, O 23, ..., to the vertices of the force polygon. From any point, f_1, on the line of action of P_1 draw two lines, $f_1 f_0$ and $f_1 f_2$, parallel to the lines O 01 and O 12; from the point f_2 in which $f_1 f_2$ meets P_2 draw $f_2 f_3$ parallel to O 23 and meeting P_3 in f_3; from f_3 draw $f_3 f_4$ parallel to O 34; and so on.

The system of lines $f_0 f_1 f_2 f_3 f_4 f_5 f_6$ parallel to the radii drawn to the vertices of the force polygon from any point, O, is called a *Funicular Polygon* of the given system of forces.

The point O the radii from which to the vertices of the force polygon determine the funicular is called the *Pole* corresponding to the funicular.

Let any other pole, O', be chosen, and from an arbitrary point, f_1', on P_1, let $f_1' f_0'$ and $f_1' f_2'$ be drawn parallel to O' 01 and O' 12, respectively; and let a new funicular, $f_0' f_1' ... f_6'$, be constructed.

Then the sides (such as $f_2 f_3$ and $f_2' f_3'$) of these polygons which reach between the lines of action of the same two forces are called corresponding sides.

Since the point f_1 may be taken anywhere on P_1 it is clear that for a given pole, O, we may construct an infinite number of funiculars of the system, but the corresponding sides of them are of course parallel.

Now observe particularly that—if the force at each vertex of a funicular of the system is resolved into two components directed along the two sides of the funicular which meet at this vertex, the components at the extremities of each side of the funicular are equal and opposite. For, suppose P_3 resolved into two components in $f_3 f_2$ and $f_3 f_4$; then these components are represented by the lines 23 O and O 34; also if P_2 is resolved into components in $f_2 f_3$ and $f_2 f_1$, these will be represented by O 23 and 12 O, respectively; thus the components in the side $f_2 f_3$ are equal and opposite.

Hence we may base the definition of a funicular on this property, thus:—*A funicular polygon of a given system of forces is* *a polygon whose vertices lie one by one on the lines of action of the given forces, and is also such that, if the force acting at each vertex is resolved into two (oblique) components along the sides of the polygon meeting in that vertex, the forces at the extremities of each side of the polygon are equal and opposite.*

It follows therefore that *if the sides of any funicular are a system of bars freely jointed to each other at the vertices, at which the forces are applied, the polygon of bars will be kept in equilibrium by the forces*, the ends of the first and last bars at which no forces are applied being pivoted. A polygon of *strings* instead of bars would not be held in equilibrium, unless the components in the sides were all *tensions*, not *pressures*. (The primary signification of *funicular* polygon—derived from *funiculus*, a little rope—has reference to strings, and has been extended in its application.)

92.] **Theorem.** *The corresponding sides of any two funiculars of a given system of forces intersect on a right line, which is parallel to that joining the poles of the two funiculars.*

At the points f_2 and f_2' let two equal forces (each P_2) be applied in opposite senses along the line $f_2 f_2'$; suppose them to act away from both of these points, as P_2 is represented in Fig. 105. Considered as acting on a rigid body, these forces are in equilibrium. Now let P_2 at f_2 be resolved into its components along $f_2 f_1$ and $f_2 f_3$. These components will be represented in magnitudes and senses by $O\,12$ and $23\,O$, respectively. Similarly, resolve P_2 at f_2' along $f_2' f_1'$ and $f_2' f_3'$; and these components will be represented by $12\,O'$ and $O'\,23$. These four components are therefore in equilibrium. Take the sum of their moments about the point of intersection of the lines $f_2 f_3$ and $f_2' f_3'$. Then, since this sum is zero, it follows that the resultant of the two components ($O\,12$ and $12\,O'$) in the lines $f_2 f_1$ and $f_1' f_2'$ must pass through the point of intersection of $f_2 f_3$ and $f_2' f_3'$; but it also passes through the point of intersection of $f_2 f_1$ and $f_2' f_1'$; therefore its line of action is the line joining these two intersections. Now this line of action is parallel to the line OO'; for, two forces represented by $O\,12$ and $12\,O'$ give a resultant represented by OO' in magnitude and sense.

Hence the corresponding sides $f_1 f_2$ and $f_1' f_2'$, $f_2 f_3$ and $f_2' f_3'$ intersect on a line parallel to OO'; similarly the sides

$f_2 f_3$ and $f_2' f_3'$, $f_3 f_4$ and $f_3' f_4'$ intersect on a line parallel to OO', which, of course, must be the same line as before. This line is LM in the figure.

Favaro (*Lezioni di Statica Grafica*, p. 409) gives a purely geometrical proof of this, depending on the property that if in two complete quadrangles five pairs of corresponding sides intersect in five points which all lie on a right line (which may be a line at infinity), the point of intersection of the sixth pair will also lie on this line.

In Fig. 105 produce $f_1 f_2$ and $f_1' f_2'$ to meet—in N, suppose; and consider the quadrangle formed by the points M, f_1, f_1', N, and also that formed by the points 01, 12, O, O'. The five pairs of corresponding sides (Mf_1, 01 O), (Mf_1', 01 O'), ($f_1 f_1'$, 1201), ($f_1 N$, 12 O), and (Nf_1', O' 12) intersect in points which lie on the line at infinity; therefore the remaining pair of sides (MN, OO') are parallel.

The general proposition—which holds equally for two quadrangles in different planes—is easily proved from the property of two triangles in perspective which will presently be given, and which is not restricted to two coplanar triangles.

93.] **Problem.** *Given one funicular of a given system of coplanar forces, to construct all funiculars of the system.*

Let the given funicular be $f_0 f_1 f_2 f_3 \ldots$. Draw any line LM in the plane of the forces; produce the sides, $f_0 f_1$, $f_1 f_2$, ..., of the given funicular to meet LM; from the point of intersection of LM and $f_0 f_1$ draw the arbitrary line $f_0' f_1'$, which meets P_1 in f_1'; join f_1' to the point of intersection of LM and $f_1 f_2$; this joining line will meet P_2 in f_2', which is the second vertex of the new funicular; join f_2' to the point of intersection of LM and $f_2 f_3$; this will give f_3'; and so on. Hence a new funicular is formed, and since the lines LM and $f_0' f_1'$ were drawn at random, an infinite number of funiculars of the system can be described in this way.

94.] **Problem.** *To construct the Resultant of a given system of coplanar forces.*

On any scale construct a force polygon 01, 12, 23,... of the given system; then the line of action of the resultant must be parallel to the side (01, 56) which closes the force polygon. Take any pole, O, and construct a funicular $f_0 f_1 f_2 \ldots$ of the system. Then the resultant must pass through the point of intersection of the extreme sides, $f_0 f_1$ and $f_5 f_6$, of the funicular. For, by resolving each force into components along the two sides of the funicular which start from the vertex at which the force may be supposed to act, these components will be mutually destroyed, with the exception of those in the extreme sides, $f_0 f_1$ and $f_5 f_6$. Hence the whole system of forces is equivalent to two forces acting in these sides, and represented in magnitudes on the scale adopted by the lines $O\,01$ and $O\,56$. The line of action of the resultant therefore passes through the intersection of the extreme sides and is parallel to the line joining 01 to 56, and the magnitude is represented by the length of this joining line, its sense being of course from 01 to 56.

The student will do well to determine by this method the resultant of the forces in Fig. 97, and also that of the parallel system in Fig. 104, verifying the results of calculation.

Cor. 1. Whatever be the path described by the pole, the point of intersection of the extreme sides of the funicular describes a fixed right line. This is the line of action of the resultant of the given system of forces.

Cor. 2. The point of intersection of any two sides of a funicular describes a fixed right line, when the pole varies in any manner. Thus the sides $f_1 f_2$ and $f_4 f_5$ will always intersect on the line of action of the resultant of the forces P_2, P_3, P_4.

95.] **Graphic Conditions of Equilibrium.** When a system of coplanar forces acting on a rigid body is in equilibrium, the forces when compounded two and two must finally reduce to two equal forces of opposite senses acting in the same right line. Since the resultant is proportional to the line required to close the force polygon, this line must be zero; hence the force polygon of the system must close up of itself. Again, since the system is finally reducible to two forces acting in the first and last sides, $f_0 f_1$ and $f_5 f_6$, of any funicular, these sides must coincide; or, in other words, the funicular must be closed.

Hence the conditions of equilibrium are—

1. The force Polygon of the system must be closed.
2. Any Funicular Polygon of the system must be closed.

Cor. 1. If any one funicular of the system is closed, every funicular of the system is closed.

Cor. 2. If the system is equivalent to a couple, the force polygon is closed, and the first and last sides of all funiculars are parallel. (Apply to the case of Art. 85.)

96.] **Problem.** *For a given system of coplanar forces find the locus of the pole of a funicular polygon two of whose sides pass each through a given point.*

It is obvious from Art. 92 that the locus is a right line parallel to that joining the two assigned points through which the two sides pass, and the line will be determined by finding the pole of any one funicular satisfying the given condition; and such a pole is easily found.

Suppose, for example, that the sides $f_0 f_1$ and $f_4 f_5$ pass each through a given point. Now these sides intersect on a given line (Art. 94), LS (Fig. 106), viz., the resultant, R_{1234}, of P_1, P_2, P_3, and P_4, parallel to the line joining 01 to 45. If, then, $f_0 f_1$ passes through the given point D, and $f_4 f_5$ through G, take any point, S, on R_{1234} and join it to D and G. Then from the vertices 01 and 45 of the force polygon draw two lines parallel, respectively, to SD and SG. These lines intersect in a point O, which is the pole of a funicular satisfying the given conditions; hence we have the locus OM which was sought. If the line joining 01 to 45 were not parallel to LS, the point S being varied, the locus of the corresponding point O would be a hyperbola. For if S_1, S_2, S_3, S_4 are any four positions of S on the right line R_{1234}, the anharmonic ratios of the pencils $D(S_1\, S_2\, S_3\, S_4)$ and $G\,(S_1\, S_2\, S_3\, S_4)$ are equal; and therefore the pencils 01 $(O_1\, O_2\, O_3\, O_4)$ and 45 $(O_1\, O_2\, O_3\, O_4)$ are also equal, which, by a well-known property, shows that the points O lie on a conic (which is a hyperbola) passing through both the points 01 and 45, its asymptotes being parallel to the lines DG and R_{1234}. The hyperbola becomes two right lines in the particular case of our problem, in which the line joining the points 01 and 45 is parallel to LS—these lines being that joining 01 to 45, and a line, OM, parallel to GD. [Deduce the locus also from Art. 36.]

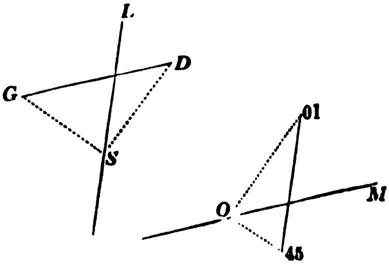


Fig. 106.

97.] **Problem.** *To represent the moment of a force about a point.*

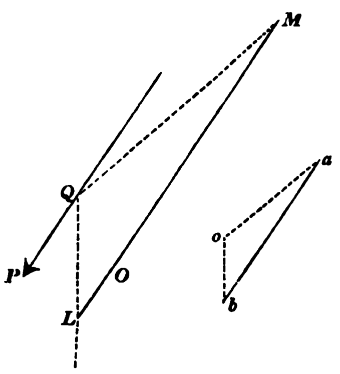


Fig. 107.

Let it be required to represent the magnitude of the moment of a force P about a point O (Fig. 107). Draw ab parallel to P and representing it on any scale.

Let o be a point taken at a unit distance from ab; draw oa and ob. Assume any point, Q, on the line of action of P, and draw QM and QL parallel to oa and ob, respectively. From O draw a line, LM, parallel to P. Then the length LM represents the moment of P about O. For, the triangles oab and QML are similar; therefore if p is the length of the perpendicular from Q on LM, we have $\frac{LM}{p} = \frac{ab}{1}$, $\therefore LM = P.p$, since ab represents P.

Hence LM is the moment on the scale adopted.

If the pole o is at a distance k units from ab, we shall have $P.p = LM \times k$.

If the unit force is ϖ, and the unit length λ, the moment of the force P about O will be $LM \times \varpi \times \frac{k}{\lambda}$; for ab will obviously be $\frac{P}{\varpi}\lambda$.

98.] **Problem.** *To represent the sum of the moments of any system of coplanar forces about a point.*

Let A (Fig. 105) be the point about which the sum of the moments of the forces is required.

The sum of their moments = the moment of their resultant about the point. Let this resultant be constructed by Art. 94, and let the moment of the resultant be constructed by last Art. Now the resultant is represented by the line joining 01 to 56 (Fig. 105), and if O is a pole assumed at any distance, k, from this line, we are to draw from any point on the resultant two lines parallel to $O\,01$ and $O\,56$, and through A a line parallel to the resultant, R.

Now the extreme sides, $f_0 f_1$ and $f_5 f_6$, of the funicular intersect in a point on R, and are parallel to the lines $O\,01$ and $O\,56$. Hence *the intercept made by the extreme sides of the funicular on a line drawn through the given point A parallel to the resultant will represent the sum of the moments of the forces about the point.*

This intercept multiplied by k will be the sum of moments.

99.] **Property of Perspective Triangles.** Two triangles, ABC and $A'B'C'$, are said to be *in perspective* when their vertices can be joined in pairs by three right lines which meet in a point. If the lines joining A to A', B to B', and C to C' meet in a point, A and A' are called *corresponding vertices*, as are also B and B', C and C'; and the sides, AB and $A'B'$, &c., which join corresponding vertices in the triangles are called *corresponding sides*.

The fundamental property of triangles in perspective is that *the points of intersection of corresponding sides lie in one right line.*

To prove this projective property it is sufficient to prove it for the simplest figure into which the two triangles can be projected. Let the line CC' be projected to infinity. Then AA' and BB' will become parallel lines; also the sides AC and BC of the first triangle will become parallel, as will $A'C'$ and $B'C'$ of the second. For the simple figure thus obtained there is no difficulty in proving the proposition.

To construct a triangle whose three sides shall pass each through a given point, and whose three vertices shall each lie on one of three concurrent lines.

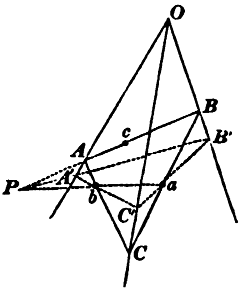


Fig. 108.

Let it be required to construct a triangle whose vertices, A, B, C, shall lie on three concurrent lines, AO, BO, CO, and whose sides shall pass through the points a, b, c, (Fig. 108). Suppose it done, and let ABC be the triangle. Take any point, C', on CO, and draw $C'a$ and $C'b$ meeting BO and AO in B' and A' respectively.

Then the triangles ABC and $A'B'C'$ are in perspective, therefore the sides AB and $A'B'$ intersect in P, a point on the line ab. Hence P is known, since it is the intersection of ab with the line $A'B'$ which is constructed by arbitrarily assuming C'. P being known, join it to c, and the vertices A and B are determined, and C follows at once. Q.E.F.

EXAMPLES ON FUNICULAR POLYGONS.

1. When three forces are in equilibrium, *any* triangle whose vertices lie on their lines of action is a funicular of the system.

(Imagine its sides to be a system of freely jointed bars. They must be in equilibrium. See end of Art. 91.)

2. A heavy rod, or beam, is supported horizontally on two smooth props at its extremities, and loaded with given weights at given points in its length; find the pressures on the props.

Suppose the line a_0a_5 (Fig. 35, p. 50) to be horizontal and to represent the loaded beam, the loads, P_1, P_2,...(including its weight among them) being applied at the points, d_1, d_2,..., and let the pressures at the props a_0 and a_5 be P_0 and P_5. Starting from any point 01 draw a vertical downward line to represent on any scale the force P_1, and let this line terminate at the point 12; from 12 draw a vertical downward line representing P_2 on the same scale, and let this line terminate at the point 23; from this point draw a vertical downward line to the point 34 to represent P_3; from 34 draw a vertical downward line to the point 45 to represent P_4.

Then from 34 we must draw a vertical *upward* line to represent the pressure P_5, and this line will terminate at the point 56, which, however, is at present unknown. The pressure P_0 will, of course, be represented by the upward line between 56 and 01.

To determine 56, assume any pole, O, and join this pole to the points 01, 12,.... Across the lines of action of the forces acting on the beam draw the lines A_0A_1, A_1A_2,... parallel to the lines $O01$, $O12$,..., and draw the closing line, A_0A_5, of the funicular polygon. Then the line through O parallel to this closing line is that joining O to the required point, 56.

3. A beam is supported horizontally at its extremities on two vertical props and loaded with given weights at given points in its length; it is required to represent the *Bending Moment* at any point of the beam.

Def. When a beam is in equilibrium under the action of any forces, the Bending Moment at any point means the sum (with their proper signs) of the moments about this point of all those forces which act at one side (either side will do) of the point.

Suppose a_0a_5 (Fig. 35, p. 50) to represent the beam, as in last example, and let P be the point about which the bending moment is required. The pressure on the prop a_0 being P_0, the bending moment at P is the sum of the moments of P_0, P_1, and P_2; and if we construct any funicular of the system this moment will, by Art. 98, be the intercept on a vertical line through P made by the extreme sides of the funicular of the forces P_0, P_1, and P_2. But these extreme sides are obviously A_0A_5 and A_2A_3. Hence the bending moment at any point P is represented by the vertical ordinate, mn, drawn through P, of any funicular polygon of the system.

Of course, if k is the distance of the pole of the assumed funicular from the vertical line which serves as the force diagram, the bending moment will be $mn \times k \times \frac{\varpi}{\lambda}$. (See end of Art. 97.)

4. Of five coplanar forces in equilibrium, given the lines of action of all, the magnitude of one, and the ratio of the magnitudes of two others; find the magnitudes of all.

Let P_1 (Fig. 109) be the force which is completely given, and let the ratio of P_2 to P_3 be given.

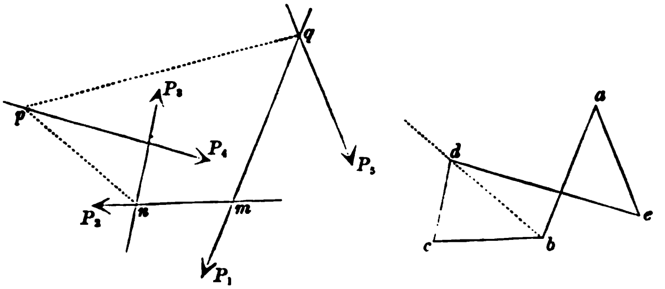


Fig. 109.

Starting with any point a, draw ab parallel and proportional to P_1; then if we draw any two lines bc, cd, parallel to the given directions P_2, P_3, and bearing to each other the given ratio, the line bd is given. Suppose bc and cd to represent P_2 and P_3; then let de and ae be drawn parallel to P_4 and P_5. It thus appears that everything would be known if any one of the points c, d, e were known.

Now, in order to get a funicular with as many known sides as possible, choose b for pole; and for further simplification start the funicular from the (given) point, m, of meeting of P_1 and P_2. We see, then, that we have to draw np parallel to the given line bd; and pq, which is parallel to be, must pass through the point of meeting of P_1 and P_5, since (for the closure of the funicular) the last side, which is parallel to ab, must be parallel to P_1, and pass through m. Now the points m and n, and therefore p, are known; hence pq is known, i.e., be is known in direction, ∴ the point e is known, and hence the force polygon is completely known.

5. Of four coplanar forces in equilibrium, given the magnitude of one and the lines of action of all; find the magnitudes of all.

6. Show how to resolve a force acting along a given line into *three* components acting each along an assigned line.

[Let the given force act in the line L, and let the other assigned lines be A, B, C. Join the point of intersection of L and A to that of B and C, and resolve the given force along the joining line and along A. Then resolve the first component along B and C.]

7. To construct for any system of coplanar forces a funicular polygon three of whose sides shall pass each through a given point.

Let the given system of forces be P_1, P_2, P_3, P_4, P_5 (Fig. 105, Art. 91), and let it be required to construct a funicular polygon which shall pass through the points D, E, F.

Consider the triangle formed by the sides f_0f_1, f_2f_3, and f_5f_6, of the funicular which pass through the three given points.

The vertex formed by the intersection of f_0f_1 and f_2f_3 lies on a given line, R_{12} (not drawn in figure), which is the resultant of P_1 and P_2 (Cor. 2, Art. 94); the vertex formed by the intersection of f_2f_3 and f_5f_6 lies on a given line, R_{345}, which is the resultant of P_3, P_4, and P_5; and the vertex formed by the intersection of f_0f_1 and f_5f_6 lies on a given line, R_{12345}, which is the resultant of P_1, P_2, P_3, P_4, and P_5.

Moreover the three lines R_{12}, R_{345}, and R_{12345} obviously meet in a point; for the resultant of $P_1, \dots P_5$ may, if we please, be constructed by first finding the resultant of P_1, P_2, and then finding the resultant of P_3, P_4, P_5.

Hence the triangle formed by the sides of the funicular which are to pass through the assigned points is one whose vertices lie on three concurrent lines and whose sides pass each through a fixed point.

Let this triangle be constructed by Art. 99. Then knowing the force diagram of the forces and drawing two lines, 01 O and 23 O say, parallel to the two sides f_0f_1 and f_2f_3, the pole O is known, and thence the whole figure.

8. Construct a funicular polygon which shall pass through three given points, two of which lie on one side of the polygon.

Ans. This side of the polygon is known, and it intersects the side passing through the remaining point in a point lying on a given line. Hence the side passing through the remaining point is known, and hence the pole of the funicular.

9. For a given system of vertical downward forces, $P_1, P_2, \dots P_{n-1}$, equilibrated by two extreme vertical upward forces, P_0, P_n, let any funicular polygon be constructed. Prove that the area of this polygon $= \frac{C}{k}$, where C is constant and k the distance of its pole from the vertical line which is the force diagram of the forces.

(The value of C is obtained by multiplying each force of the system by half the product of the distances between its line of action and the lines of action of the extreme forces, and adding all such products together, and multiplying the result by $\frac{\lambda}{\varpi}$. See end of Art. 97.)

10. A uniform beam is supported at its extremities on two vertical props; find the bending moment at any point in it.

Ans. If y is the distance of the point from one extremity, the bending moment is $W\frac{y(l-y)}{2l}$, where W is the weight of the beam.

11. In the last example what is the curve of bending moment?

Ans. A parabola passing through the ends of the beam, its vertex lying on the vertical line through the middle of the beam at a distance $\frac{l}{8}$ from the beam. (The bending moment at any point is the product of W and the vertical distance of the point from the parabola.)

12. For any assigned system of forces, construct a funicular polygon such that if it were actually a string or a system of jointed bars kept in equilibrium (its two extremities being fixed) by the given forces, the sum of the squares of the tensions or pressures in its sides would be a minimum.

[Choose for pole the centroid of the vertices of the force-polygon.]

100.] Application of Statics to Geometry. Theorems in geometry are often deducible from simple considerations with regard to the circumstances of given systems of forces. Thus, Ptolemy's theorem that *the rectangle under the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the rectangles under the opposite pairs of sides* follows (see example 13, p. 19) from the fact that, if $ABCD$ is such a quadrilateral, two forces acting in AB and AD respectively proportional to the opposite sides, DC and BC, give a resultant in AC proportional to the other diagonal, BD. Expressing the fact that along the diameter of the circle the component of this resultant is equal to the sum of the components of the forces in AB and AD, we have the theorem in question.

Again, the well-known result that *the feet of the perpendiculars on the sides of a triangle from any point on the circumscribing circle are collinear* follows from example 7, p. 133, because, since the isogonal conjugate of any point on this circle is at infinity, the algebraic sum of the three multiples placed at A, B, C is zero, i.e., the area of the triangle $PQR = 0$.

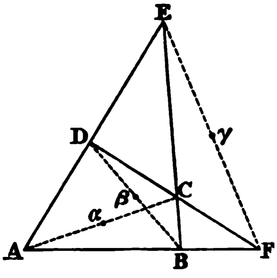


Fig. 110.

Also the theorem that *the middle points of the diagonals of any complete quadrilateral all lie on a right line* follows by taking the following system of forces, supposed acting on a rigid body: two forces represented by DA and DC in magnitudes and senses, and two represented by BA and BC (Fig. 110).

Now the resultant of the first pair passes through α, the middle point of AC; so does the resultant of the second pair; therefore the resultant of the four forces passes through α. Also the resultant of DA and BA passes through β, the middle point of BD; so does the resultant of DC and BC; hence the resultant of the four forces also passes through β. We shall now show that it passes through γ, the middle point of EF. For this purpose introduce a force ED and a force DE which, destroying each other, do not alter the given system. Introduce also forces CE, EC; CF, FC; FB, BF. Hence the given system is equivalent to forces EA, FA; DF, DE; BE, BF; EC, FC; and it is obvious that the resultant of each of these pairs passes through γ; hence the resultant of the whole system passes through γ.

Now as the resultant of the given system acts in a right line, and as α, β, γ have been independently shown to be points on this resultant, these points are collinear.—Q. E. D.

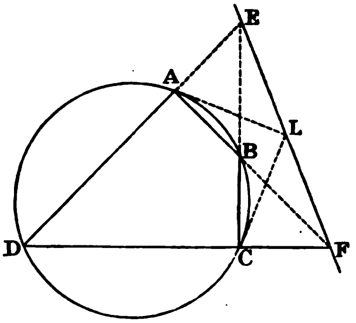


Fig. 111.

The *method* of this example ought to be kept in view for application to similar cases. Of course the whole difficulty in such cases consists in discovering the appropriate force system.

Another example is the following: *if a quadrilateral is inscribed in a circle, the tangents at the opposite vertices intersect on the third diagonal.* Imagine a system of forces acting in cyclical order along the sides, each force being *inversely* proportional to the length of the side in which it acts (Fig. 111).

Now, example 12, p. 19, the resultant of $\frac{\kappa}{DA}$ and $\frac{k}{AB}$ acts in the tangent at A; the resultant of $\frac{k}{BC}$ and $\frac{k}{CD}$ acts in the tangent at C; and if these tangents intersect in L, the resultant of the whole system passes through L. Similarly if the tangents at

B and D intersect in M, the resultant passes through M. But the resultant acts in the line EF: for the resultant of $\frac{k}{DA}$ and $\frac{k}{BC}$ passes through E, and it also passes through F, because these forces are easily seen to have equal and opposite moments about F; and as the resultant of the forces $\frac{k}{AB}$ and $\frac{k}{CD}$ passes through F, the resultant of the whole system passes through F; and similarly through E. Hence EF is the line of action of the resultant of the system and must contain the points L and M.

The *property of perspective triangles* (Art. 99) can also be deduced either from the theorem of Art. 92, or from the very elements of Statics. For, along the three concurrent lines OA, OB, OC, (Fig. 108) let three equilibrating forces act. Then since (example 1, Art. 99) each of the two triangles ABC, $A'B'C'$ is a funicular, their corresponding sides intersect in three collinear points. The result may also be deduced thus: let the system of equilibrating forces in OA, OB, OC be P_1, P_2, P_3; let P_1 be applied at A and $-P_1$ at A'; P_2 at B and $-P_2$ at B'; P_3 at C and $-P_3$ at C'; resolve each force into two components along the two sides which meet in the vertex at which the force is applied; let the components of P_3 be a, β in CB, CA, respectively; let those of $-P_3$ at C' be a', β' along $C'B'$, $C'A'$, respectively; those of P_2 being $-a$, γ; etc. Then since the forces P_2, $-P_2$, P_3, $-P_3$ are in equilibrium, their components $-a$, γ, $-a'$, γ', a, β, a', β' are equilibrium, that is, γ, γ', β, β' are in equilibrium, $\therefore$ the resultant of γ, γ' is equal and opposite to that of β, β', $\therefore$ each must act in the line Pb. Similarly by taking the forces P_2, $-P_2$, P_1, $-P_1$ we find that a, a', β, β' must be in equilibrium; but the resultant of a, a' acts through the point a, $\therefore$ aPb must be a right line.

Finally, Pascal's Theorem, that *the intersections of the opposite sides of a hexagon inscribed in a circle lie in a right line* is easily exhibited as a case of the funicular property in Art. 92.

Let the lines DA, EB, FC (Fig. 112) be lines of action of three forces, P, Q, R such that if P is resolved at A into two components along AB, AF, or into two at D along DC, DE; if Q is resolved into two at B along BA, BC, or into two at E along ED, EF; and if R is resolved at C along CB, CD, or at F along FE, FA, the two components thus obtained along any side are equal and opposite. Obviously such conditions are consistent, on account of the equality of angles in the same segment of a circle. Now if P, Q, R are applied at A, B, C, by the nature of the case a polygon $FABCD$ of jointed bars pivoted at F and D would be kept in equilibrium, i. e., this is a funicular of the forces.

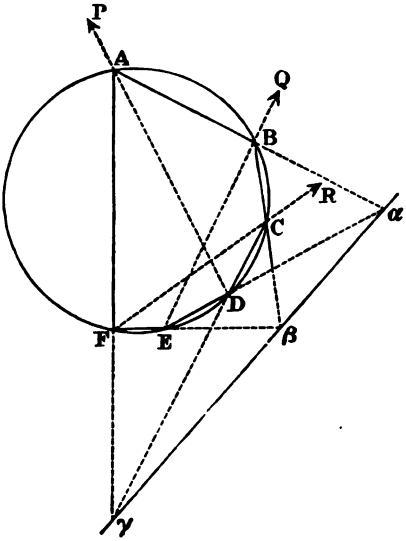


Fig. 112.

Again, let P, Q, R be applied at D, E, F, to a polygon $CDEFA$ of jointed bars pivoted at C and A. This polygon would be in equilibrium, and a funicular of the forces. The two polygons, therefore, are two funiculars of the same forces, and therefore the intersections, α, β, γ, of their corresponding sides (AB, DE), (BC, EF), (CD, FA) are collinear.—Q. E. D.

The ratios $P : Q : R = \sin A \sin D : \sin B \sin E : \sin C \sin F$.

EXAMPLE.

If a circle be described about any triangle, ABC, and the Symmedian chords of the circle through A, B, and C be drawn, prove that the length of each chord is inversely proportional to the sine of the angle between the other two. (Mr. McCay, F.T.C.D.) (See example 5, Art. 88.)

101.] **Astatic Equilibrium.** When any number of forces, P_1, P_2, ..., acting at points, A_1, A_2, ..., in a body keep this body in equilibrium, these forces will not, in general, continue to preserve equilibrium when the body is displaced in any manner, each force still retaining its magnitude, direction, and point of application in the body. If for all displacements of the body the forces continue to preserve equilibrium, the body is said to be in *astatic* equilibrium.

The simplest example of astatic equilibrium is furnished by a heavy body suspended by a vertical string attached at its centre of gravity. Here the system of forces consists of the weights of the particles of the body and the tension of the string; and however the body may be displaced about its centre of gravity, all these forces will retain their individual magnitudes, directions, and points of application, and the body will remain at rest.

Again, a system of two equal reversed magnets rigidly connected by an axis through their centres is astatic for displacements round this axis.

When a system of forces applied to a body is not in equilibrium, it happens that in certain cases this system can be astatically equilibrated by a single applied force; i. e., in all displacements which the body can receive, each force acting on it with invariable magnitude, direction, and point of application, it may be possible to equilibrate the system by one force of constant magnitude, direction, and point of application.

It is evident that this is always the case for a system of parallel forces. A single force equal and opposite to their resultant, applied at their centre, will astatically equilibrate them.

Into the general discussion of astatic equilibrium we do not at present enter. Suffice it to say that a system of (non-coplanar) forces must in general be astatically equilibrated by *three* forces; and if the forces are all parallel to one plane, by *two*. When (as in the present chapter) the forces are all coplanar we shall prove that for displacements of their points of application in their plane the system can be astatically equilibrated by a single force.

In this case it is clear that instead of considering the body to which they are applied as displaced, we may consider the body fixed and each force rotated in a fixed sense round its point of application through a constant angle—a motion of translation of the body or points having obviously no effect on the system of forces.

We shall now prove that—*if all the forces in a coplanar system are rotated in the same sense, through the same angle, in their plane, round their points of application, their resultant* (unaltered in magnitude, of course) *passes through a fixed point in the body.*

Let two forces, P and Q, act at two fixed points, A and B, (Fig. 113) in the directions OA and OB, O being the point of intersection of their lines of action; and let the forces be turned in the same sense round A and B through the same angle, so that the point of intersection of their new lines of action is O'. Now, since $\angle OAO' = \angle OBO'$, a circle described through A, B, and O will pass through O', and the angle $AO'B$, between P and Q when they are turned round, is equal to the original angle, AOB, between them. Also, the forces being unaltered in magnitude, it follows that the angles which the resultant at O' makes with them are the same as the angles which it makes with P and Q at O. If, then, OC is the direction of the resultant at O, $O'C$ must be the direction of this resultant at O'. Hence, the resultant of P and Q passes through the fixed point C. In exactly the same way it is proved that the resultant of three forces passes through a fixed point when the forces are turned round their fixed points of application through a constant angle; and so on for any number of forces.

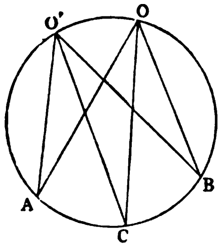


Fig. 113.

This point may be called the *astatic centre* of the system of forces*.

102.] **To find the Astatic Centre of a System of Coplanar Forces.** Taking an arbitrary origin and arbitrary axes, the point required lies on the resultant whose equation is (Art. 88)

$$\alpha\Sigma Y - \beta\Sigma X - G = 0, \qquad (1)$$

(α, β) being the running co-ordinates.

Now, if the force P_1 acting at the point (x_1, y_1) is turned round in the plane of xy through an angle ω, its component X_1 becomes

$$P_1 \cos(\theta_1 + \omega),$$

where θ_1 is the original angle made with the axis of x by P_1, or $X_1 \cos\omega - Y_1 \sin\omega$; Y_1 becomes $X_1 \sin\omega + Y_1 \cos\omega$; and $Y_1x_1 - X_1y_1$ becomes $(Y_1x_1 - X_1y_1)\cos\omega + (X_1x_1 + Y_1y_1)\sin\omega$.

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* Of course it is understood throughout this discussion and in the examples at the end of this chapter that the displacements of the body or forces are always supposed to take place in the plane of the forces.

$$\left.\begin{array}{lll} \text{Hence, } \Sigma X & \text{becomes} & \cos\omega \,.\, \Sigma X - \sin\omega \,.\, \Sigma Y, \\ \Sigma Y & \quad ,, & \sin\omega \,.\, \Sigma X + \cos\omega \,.\, \Sigma Y, \\ G & \quad ,, & G\cos\omega + \Gamma\sin\omega, \end{array}\right\} \qquad (A)$$

where $\Gamma \equiv \Sigma(Xx + Yy)$. This quantity is called the Virial of the forces.

The equation of the new resultant is, therefore,

$$(\alpha\Sigma Y - \beta\Sigma X - G)\cos\omega + (\alpha\Sigma X + \beta\Sigma Y - \Gamma)\sin\omega = 0, \qquad (2)$$

and the astatic centre of the system of forces is the intersection of the lines given by equations (1) and (2). This point may evidently be determined by (1) and by the equation

$$\alpha\Sigma X + \beta\Sigma Y - \Gamma = 0. \qquad (3)$$

Hence for the co-ordinates of the astatic centre we have

$$\alpha = \frac{\Gamma\Sigma X + G\Sigma Y}{R^2}, \quad \beta = \frac{\Gamma\Sigma Y - G\Sigma X}{R^2}. \qquad (4)$$

If the astatic centre were the origin, α and β would be each $= 0$, and G would $= 0$, since the point is on the resultant (Art. 82). Hence for the centre of the forces we have

$$G = 0, \quad \Gamma = 0. \qquad (5)$$

If the co-ordinates of A, the point of application of a force, P, (Fig. 114), with respect to rectangular axes, Ox and Oy, are x and y, the quantity $Xx + Yy$ is equal to $P\ (x\cos\theta + y\sin\theta)$, θ being the angle which P makes with Ox. Now if OM is x, and AM is y, it is evident that $x\cos\theta + y\sin\theta = AN$, N being the foot of the perpendicular from O on the line of action of P. Denoting AN by q, we have, then, for the Virial

$$\Gamma = \Sigma(Pq).$$

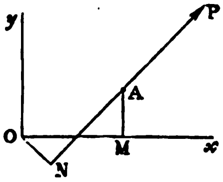

Fig. 114.

Hence, *if any number of coplanar forces be turned each round a fixed point of application through an arbitrary but common angle, there exists a point in the plane of the forces such that both the Virial and the sum of the moments of the forces about it continue to vanish for all displacements.*

It is easy to see that if AN be the sense in which P acts, the sign of the product Pq will be changed.

The value of Γ with respect to axes through a point (α, β) parallel to Ox and Oy is evidently $\Sigma\{X(x-\alpha) + Y(y-\beta)\}$, or $\Gamma - \alpha\Sigma X - \beta\Sigma Y$. Hence the locus of points for which this quantity vanishes is given by equation (3), which denotes a right line passing through the astatic centre, and evidently perpendicular to the resultant.

103.] **Theorem.** *If any number of coplanar forces are in equilibrium, and if the forces be turned, each round a fixed point, in the same sense through any common angle, the new system is equivalent to a couple.*

For, from equations (A) of Art. 102, it appears that if $\Sigma X = 0$ and $\Sigma Y = 0$ before the rotation, they will be zero after it; hence the new system has no resultant of translation, and it must, therefore, be a couple. Now, since by hypothesis $G = 0$, the axis of the new couple is by equations (A) equal to

$$\Gamma \sin \omega.$$

We see, then, that the system of forces will remain in equilibrium, whatever be the angle through which they are turned, if

$$\Gamma = 0.$$

EXAMPLES.

1. If a system of coplanar forces applied at fixed points is in equilibrium, the co-ordinates of the astatic centre become indeterminate. Explain this.

Ans. In this case the system must be astatically equilibrated by two equal and opposite parallel forces.

2. In the last case show how to find an astatically equilibrating couple for the system.

Ans. Take the astatic centre of any number of the forces, and also the astatic centre of the remaining forces. These will be the points of application of the forces of the required couple (whose moment, of course, varies with the displacement of the body or forces), and the forces of the couples are equal to the resultants of the two partial sets.

3. Three forces are applied at the middle points of the sides of a triangle, *ABC*, perpendicular to the sides and proportional to them respectively; find a couple which will astatically equilibrate them.

Ans. A couple one of whose forces is applied at the middle point of any one side, *AB*, and the other applied at the point of intersection of a parallel to *AB* drawn through *C* with the perpendicular to *AB* at its middle point.

4. When a system of coplanar forces in equilibrium continues in equilibrium for all displacements in the plane of the forces, show that the astatic centre of any number of them must be coincident with that of the remainder.

CHAPTER VI.

APPLICATION OF THE CONDITIONS OF EQUILIBRIUM OF A BODY.

104.] **Condition of Equilibrium of a Body under the Action of two Forces in a Plane.** *If two forces maintain a body in equilibrium, they must be equal and opposite in the same right line.*

For, take moments round any point on the line of action of one of them, P. The sum of the moments must (Art. 86) be $= 0$. Hence the other force, Q, must pass through the assumed point. Again, take any other point on P, and take moments round it. The sum must be $= 0$, and Q must, therefore, pass through this point. Hence P and Q act in the same line. Now their sum must $= 0$ (Art. 86). Therefore P and Q are equal and opposite.—Q. E. D.

105.] **Condition of Equilibrium of a Body under the Action of three Forces in one Plane.** *If three forces maintain a body in equilibrium, their lines of action must meet in a point, or be parallel.*

For, take moments round the point of intersection of two of them, P and Q. The sum must (Art. 86) $= 0$; therefore, either the third force, R, is zero, or it passes through the intersection of P and Q. If R is not $= 0$, it must pass through this point.

The three forces may then be supposed to act at this point, and to keep it at rest. Hence, each force must be equal and opposite to the resultant of the other two; and if the angles between them in pairs be p, q, r, the forces must satisfy the conditions

$$P : Q : R = \sin p : \sin q : \sin r. \qquad (\beta)$$

If two of them are parallel, the third must be parallel to them and equal and directly opposed to their resultant.

vanishes is given by equation (3), which denotes a right line passing through the astatic centre, and evidently perpendicular to the resultant.

103.] **Theorem.** *If any number of coplanar forces are in equilibrium, and if the forces be turned, each round a fixed point, in the same sense through any common angle, the new system is equivalent to a couple.*

For, from equations (A) of Art. 102, it appears that if $\Sigma X = 0$ and $\Sigma Y = 0$ before the rotation, they will be zero after it; hence the new system has no resultant of translation, and it must, therefore, be a couple. Now, since by hypothesis $G = 0$, the axis of the new couple is by equations (A) equal to

$$\Gamma \sin \omega.$$

We see, then, that the system of forces will remain in equilibrium, whatever be the angle through which they are turned, if

$$\Gamma = 0.$$

EXAMPLES.

1. If a system of coplanar forces applied at fixed points is in equilibrium, the co-ordinates of the astatic centre become indeterminate. Explain this.

Ans. In this case the system must be astatically equilibrated by two equal and opposite parallel forces.

2. In the last case show how to find an astatically equilibrating couple for the system.

Ans. Take the astatic centre of any number of the forces, and also the astatic centre of the remaining forces. These will be the points of application of the forces of the required couple (whose moment, of course, varies with the displacement of the body or forces), and the forces of the couples are equal to the resultants of the two partial sets.

3. Three forces are applied at the middle points of the sides of a triangle, ABC, perpendicular to the sides and proportional to them respectively; find a couple which will astatically equilibrate them.

Ans. A couple one of whose forces is applied at the middle point of any one side, AB, and the other applied at the point of intersection of a parallel to AB drawn through C with the perpendicular to AB at its middle point.

4. When a system of coplanar forces in equilibrium continues in equilibrium for all displacements in the plane of the forces, show that the astatic centre of any number of them must be coincident with that of the remainder.

CHAPTER VI.

APPLICATION OF THE CONDITIONS OF EQUILIBRIUM OF A BODY.

104.] **Condition of Equilibrium of a Body under the Action of two Forces in a Plane.** *If two forces maintain a body in equilibrium, they must be equal and opposite in the same right line.*

For, take moments round any point on the line of action of one of them, P. The sum of the moments must (Art. 86) be $= 0$. Hence the other force, Q, must pass through the assumed point. Again, take any other point on P, and take moments round it. The sum must be $= 0$, and Q must, therefore, pass through this point. Hence P and Q act in the same line. Now their sum must $= 0$ (Art. 86). Therefore P and Q are equal and opposite.—Q. E. D.

105.] **Condition of Equilibrium of a Body under the Action of three Forces in one Plane.** *If three forces maintain a body in equilibrium, their lines of action must meet in a point, or be parallel.*

For, take moments round the point of intersection of two of them, P and Q. The sum must (Art. 86) $= 0$; therefore, either the third force, R, is zero, or it passes through the intersection of P and Q. If R is not $= 0$, it must pass through this point.

The three forces may then be supposed to act at this point, and to keep it at rest. Hence, each force must be equal and opposite to the resultant of the other two; and if the angles between them in pairs be p, q, r, the forces must satisfy the conditions

$$P : Q : R = \sin p : \sin q : \sin r. \qquad (\beta)$$

If two of them are parallel, the third must be parallel to them and equal and directly opposed to their resultant.

EXAMPLES.

1. Three forces, P, Q, R (Fig. 115), act at the middle points of the sides of a triangular plate, each force being perpendicular and proportional to the side at which it acts. If the forces all act inwards, or all outwards, they are in equilibrium. For (α) they satisfy the first condition of equilibrium of three forces, namely, that of meeting in a point (Art. 105); and (β) they are proportional to the sines of the angles between them in pairs, since

$$P:Q:R = a:b:c = \sin A:\sin B:\sin C$$
$$= \sin QOR:\sin ROP:\sin POQ.$$

They, therefore, satisfy both of the conditions of Art. 105.

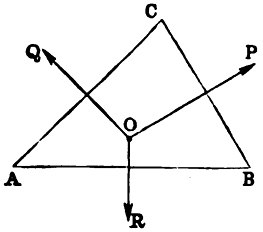

Fig. 115.

In exactly the same way it is proved that if three forces act perpendicularly to the sides of a triangle, and be proportional to them, they will be in equilibrium, provided that they pass through any common point, and all act outwards or all inwards.

2. Three forces acting along the perpendiculars of a triangle keep it at rest; find the relations between them.

They satisfy the first condition of equilibrium, namely, that of meeting in a point. Then if the forces perpendicular to the sides a, b, c, be P, Q, R, respectively, the relations (β) of Art. 105 give

$$P:Q:R = \sin A:\sin B:\sin C = a:b:c,$$

as might have been concluded from the remark at the end of the last example.

3. Three forces acting along the bisectors of the angles of a triangle, all either from or towards the vertices, keep it at rest; find the relations between them.

The forces evidently satisfy the condition of meeting in a point.

Let P, Q, R, be the forces in the bisectors of A, B, C, respectively. Then the angle between P and Q is easily seen to be $\pi - \frac{A+B}{2}$.

Hence $$P:Q:R = \cos\frac{A}{2}:\cos\frac{B}{2}:\cos\frac{C}{2}.$$

4. Three forces acting in the bisectors of the sides of a triangle drawn from the opposite vertices maintain equilibrium; find the relations between them.

They satisfy the first condition.

Let the lengths of the bisectors of the sides a, b, c (Fig. 116) be β_1, β_2, and β_3, and let p and q be the perpendiculars from C on P and Q.

Take moments round C for the equilibrium of the forces. Then

$$Pp = Qq. \qquad (1)$$

(The moments of P and Q with respect to C have opposite signs, since Q tends to turn the body round C in the sense of watch-hand rotation, while P tends to turn it in the opposite sense.)

Again, $p\beta_1 = q\beta_2$, (2)
each side of this equation being the area of the triangle. Divide the sides of (1) by the corresponding sides of (2).

Then $$\frac{P}{\beta_1} = \frac{Q}{\beta_2}.$$

Hence $P : Q : R = \beta_1 : \beta_2 : \beta_3$, or the forces are proportional to the bisectors.

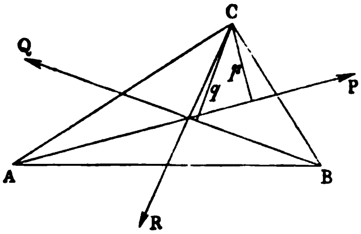

Fig. 116.

5. At the middle points of the sides of any *indeformable* polygon (Fig. 117) forces act perpendicularly to the sides, each force being proportional to the side at which it acts. If the forces all act inwards or outwards, they form a system in equilibrium.

For (example 1) the resultant of P_1 and P_2 is a force acting at the middle point of AC, perpendicular and proportional to AC. Again, this force and P_3 may be replaced by a force acting at the middle point of AD, perpendicular and proportional to AD.

Replacing the given forces in this manner, the result follows by example 1.

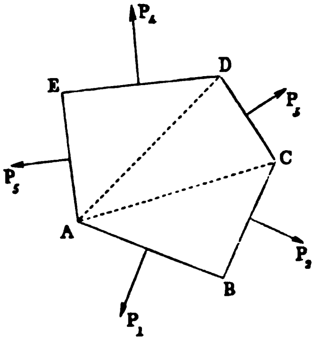

Fig. 117.

6. If from any point perpendiculars be drawn to the sides of a polygon, and forces act along these perpendiculars, either all inwards or all outwards, each force being proportional to the side to which it is perpendicular, the system is in equilibrium.

This follows, exactly as in the last example, by dividing the polygon into triangles, and attending to the remark at the end of example 1.

7. From any point, O, inside (or outside) a triangle, ABC (Fig. 118), are let fall perpendiculars, $O\alpha$, $O\beta$, $O\gamma$, on the three sides. At the points α, β, γ, are applied forces P, Q, R, each of which is proportional and perpendicular to the side at which it acts. The forces are then all turned round their points of application in the same sense, so as to make equal angles with the perpendiculars $O\alpha$, $O\beta$, and $O\gamma$. Show that in this latter case the resultant of the system of forces is a couple whose moment is proportional to the square root of the area of the triangle $A'B'C'$, enclosed by their lines of action.

(The forces act all outwards or all inwards.)

Let the sides of ABC be a, b, c, and let $P = ka$, $Q = kb$, $R = kc$, k being a constant coefficient.

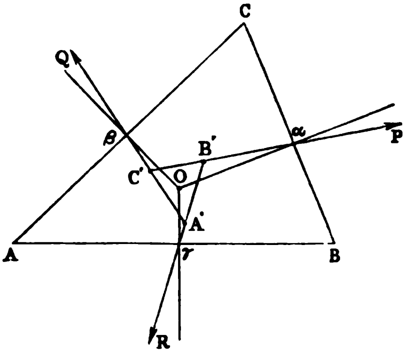

Fig. 118.

Let θ be the angle, OaB', between P and the perpendicular Oa. Then

$$\theta = O\beta C' = O\gamma A'.$$

Replace P by two components, one along BC and the other perpendicular to it. Similarly, replace Q and R. Then the perpendicular components are $ka \cos\theta$, $kb \cos\theta$, and $kc \cos\theta$; and since they meet in a point, O, and are proportional to the sides at which they act, they are in equilibrium (example 1). Hence the forces are equivalent to three, $ka \sin\theta$, $kb \sin\theta$, and $kc \sin\theta$, acting along the sides of ABC in cyclical order, and therefore, their equivalent is a couple $= 2k\Delta \sin\theta$, Δ denoting the area of the triangle ABC. (See Art. 103, p. 152.) Now the triangle $A'B'C'$ is similar to ABC. For, since the angles OaB and $O\gamma B$ are right, and the angles OaB' and $O\gamma B'$ are equal, a circle will go round the points $OB'aB\gamma$. Hence $\angle\gamma Oa = \angle\gamma B'a$; therefore their supplements, B and B', are equal. Similarly, $A = A'$, and $C = C'$.

Again, the side $A'B' = AB \,.\, \sin\theta$. For, in the circle round $\gamma OB'aB$, $\gamma B'$ is a chord making an angle θ with a chord γO, and an angle $\frac{\pi}{2} - \theta$ with the perpendicular chord, γB. Therefore

$$\gamma B' = \gamma O \,.\, \cos\theta + \gamma B \,.\, \sin\theta. \tag{1}$$

Similarly, in the circle round $\gamma A' O\beta A$, we have

$$\gamma A' = \gamma O \,.\, \cos\theta - \gamma A \,.\, \sin\theta. \tag{2}$$

Subtracting (2) from (1) we have

$$A'B' = (\gamma B + \gamma A) \,.\, \sin\theta = AB \,.\, \sin\theta.$$

Now if Δ' be the area of $A'B'C'$,

$$\frac{\Delta'}{\Delta} = \left(\frac{A'B'}{AB}\right)^2 = \sin^2\theta;$$

$$\therefore \quad \sin\theta = \sqrt{\frac{\Delta'}{\Delta}},$$

and therefore the moment of the forces $= 2k\sqrt{\Delta\Delta'}$.

8. If the triangle be replaced by a polygon of any number of sides, prove that the equivalent of the forces is a couple whose moment is proportional to the square root of the area of the (similar) polygon enclosed by their lines of action.

9. A heavy beam, AB (Fig. 119), rests against a smooth horizontal plane, CA, and a smooth vertical wall, CB, the lower extremity, A, being attached to a cord which passes over a smooth pulley at C, and sustains a given weight, P. Find the position of equilibrium, and the pressures on the plane and wall.

Let θ be the inclination of the beam to the horizon in the position of equilibrium; let $W =$ weight of the beam: and let the centre of gravity, G, divide the beam into two portions, $AG = a$, and $BG = b$.

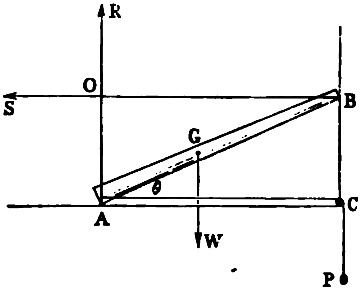

Fig. 119.

Now, the reactions, R and S, of the wall and plane are normal to these surfaces; and since they are both unknown, we shall obtain an equation for θ which will contain neither of them, by taking moments about O, their point of intersection. Hence, since the force P acts on the beam along AC, and tends to turn it in a sense opposite to that in which W tends to turn it round O, we have

$$P(a+b)\sin\theta - Wa\cos\theta = 0,$$

$$\therefore \quad \tan\theta = \frac{Wa}{P(a+b)}. \tag{1}$$

Again, resolving forces vertically, we have

$$R = W. \tag{2}$$

And resolving horizontally, $S = P$. (3)

10. If the beam rest, as in the last example, against a smooth vertical and a smooth horizontal plane, and a cord be attached firmly to the point C, and to a point in the beam, find the limit to the position of this latter point consistent with equilibrium.

Let Fig. 120 represent the beam in any position, and let m be the middle point of the beam. Suppose the cord attached to C, and to a point, n, in the upper half of the beam. Then the forces acting on the beam are W, T (the tension of the cord nC), R, and S. Let p be the point of intersection of W and T. Now, the resultant of W and T must, for equilibrium, be equal and opposite to the resultant of R and S; hence the resultant of R and S must act in the line Op; but this line is not between the lines of action of T

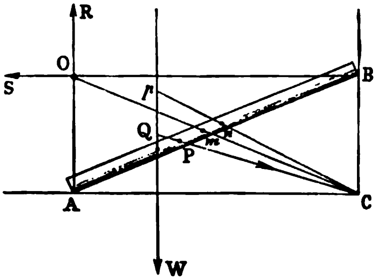

Fig. 120.

and W, that is, inside the angle WpC; therefore the resultant of R and S cannot be equal and opposite to that of W and T with such a position of the cord, and, therefore, equilibrium is impossible, no matter what the inclination of the beam may be. Hence, in order that equilibrium may be possible, the cord must be attached to some point, such as P, between A and m.

11. In the last example, given the point of attachment of the cord, find the tension in it.

It is easy to see that if P, the point of attachment, be given, and also l, the length of the cord, CP, the position of the beam, is given. For, if $\theta = \angle BAC$, we have

$$l^2 = BP^2 . \cos^2\theta + AP^2 . \sin^2\theta,$$

an equation which determines θ.

The angle PCA is also known. Denote it by ϕ. To determine T, the tension of the cord, without bringing R and S into our equation, take moments round O, their intersection. Hence, a and b being the segments of the beam made by the centre of gravity, we have

$$Wa \cos\theta = T . OC \sin OCP = T . (a+b) \sin(\theta-\phi),$$

$$\therefore \quad T = W \cdot \frac{a\cos\theta}{(a+b)\sin(\theta-\phi)}.$$

It will be a good exercise for the student to find R, S, and T by graphic statics. [See example 5, p. 143.]

Note. If $\theta = \phi$, $T = \infty$. In this case the cord is attached to m, the middle point of the beam, and therefore its direction always passes through O, the intersection of R and S. Now, it is easy to see that in this case the conditions of equilibrium are theoretically satisfied, because the resultant of T and W acts along T, whose direction passes through O. But if $\phi > \theta$, no value of T can even theoretically satisfy the conditions (see last example).

12. ABC is any triangle, of which C is the vertex. It is acted on by the forces CA, CB, and AB. Prove that it will be kept in equilibrium by a force equal to $2\,BC$, acting parallel to BC, at the middle point of AB.

13. In example 12, it is clear that two positions of equilibrium of the beam are a vertical and a horizontal position; explain why these positions are not given by the equation (1) which determines the position of equilibrium.

14. Explain why the proof in example 5 would not hold for a polygon formed of bars freely jointed together and therefore capable of turning about the joints.

106.] **Action of a Hinge or Joint.** Among the internal forces of a system, the action of a cylindrical axis is one of frequent occurrence. If the axis is smooth, the reaction between two bars or beams connected by it consists of a single force passing through its centre. For, let PQS (Fig. 121) represent a section of the joint connecting two beams: then, since their surfaces are in contact, either throughout the whole of the circumference or a part of it, there will be (since the joint is smooth) *normal* reactions at the points of contact, P, Q, ... Now, since all these pass through the centre of the circle, they have a single resultant through this point. Consequently, the action in this case consists of a single force *through the centre of the joint.*

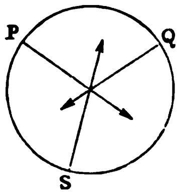

Fig. 121.

But, if the joint be rough, the reactions at the points of contact will not be normal, that is, their lines of action will not meet in a point, and, therefore, they may reduce to a couple, or to a single force. When slipping is about to ensue at the joint, it is easy to see that the total resistances at the points of contact envelop a circle (or rather a cylinder). For, at any point, P, of contact (Fig. 122), draw PR, making the angle of friction, λ, with the normal, PC, to the surface of contact. The perpendicular from C, the centre of the joint, is equal to $PC \cdot \sin\lambda$, and is, therefore, constant. Hence PR envelops a circle whose radius $= PC \cdot \sin\lambda$.

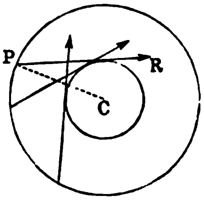

Fig. 122.

If $PC = a$, and ds is the element of the surface of contact at P, it is evident that the sum of the moments of the reactions about C is (R being the reaction per unit of surface)

$$a \sin\lambda \int R ds.$$

As an example, let us consider the equilibrium of two equal bars which are connected by a joint, C, and rest on a smooth horizontal cylinder, in a vertical plane at right angles to the axis of the cylinder.

Firstly, let the joint be rough, and suppose the contact to be complete all over its surface: then it is clear that such a position as that represented in Fig. 123 is a possible position of equilibrium if the joint is sufficiently

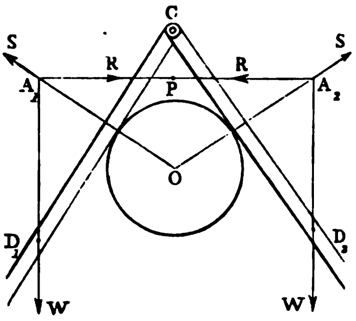

Fig. 123.

rough. Let Fig. 124 represent an enlarged view of the circle which is enveloped by the total resistances at the various points of the surface of contact at the hinge, C. Then, if the total resistances at the lower portion of the joint be considerably greater than those at the upper portion, it is possible that the resultant of the whole set may be a horizontal force, R, acting through a point, P, below the joint.

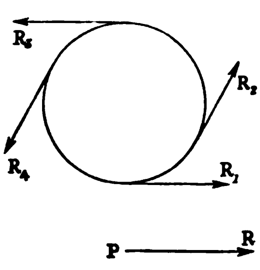

Fig. 124.

In the position of equilibrium of the bars represented in Fig. 123 the weight, W, of the bar CD_1, and the normal reaction, S, of the smooth cylinder, meet in a point A_1, through which point the force produced by the action of the other bar must pass. In the same way the action of the bar CD_1 on CD_2 must pass through the point A_2. Hence the resultant action of each bar on the other must be directed in the line $A_1 A_2$; and we have seen that if the contact along the joint extend over its surface, this is a possible line of action, though it does not intersect the joint.

Secondly, let the joint be rough, and let the contact take place at only one point, N (Fig. 125). Suppose the joint to consist of a pin, BN, which forms part of the bar CD_2 (Fig. 123), and let this fit loosely into the bar CD_1. It is clear, then, that the action between the bars consists of a single force, R, acting at N, and making the angle of friction, λ, with the radius CN, if slipping is about to take place. As before, this force must pass through the points A_1, A_2.

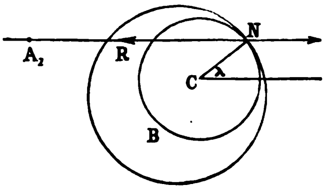

Fig. 125.

In this case, then, the point of contact of the bars is constructed by drawing a radius, CN, of the cylindrical axis constituting the joint, inclined to the horizon (since $A_1 A_2$ is horizontal) at the angle of friction.

Thirdly, let the joint be smooth. In this case the bars must assume such a position that the line $A_1 A_2$ passes through the centre of the joint; and this position is practically the same as that in the last case, because since the dimensions of the joint are negligible compared with those of the bars, the line of resistance *RN* (Fig. 125) may be supposed to pass through the centre, *C*, of the joint.

A similar explanation is to be given in the case of two equal beams rigidly connected, and forming one piece, the system resting, as in the previous example, on a smooth cylinder. In this case the beams can take only one position, which must be a position of equilibrium, and the action between them must accommodate itself to the geometrical necessity of the figure. (In the following figure the cylinder is not drawn.) If we consider the equilibrium of one of the beams, *CD* (Fig. 126), by itself, we shall have to supply to it whatever force is actually produced upon it by the other beam. Now, if *BC* is the section along which the system is considered as divided by the removal of the second beam, it is clear that the internal forces in the neighbourhood of *B* tend to tear the beams apart, if *A* is below the section *BC*, while those about *C* tend to press the beams more closely together. Hence the action of the second beam on *CD* consists of a number of forces whose horizontal components near *B* act from left to right, as the force *BF*, and whose horizontal components near *C* act from right to left, as the force *CF'*. If, therefore, the forces near *B* are greater than those near *C*, the resultant of the whole system will consist of a horizontal force, *AR*, acting outside the section *CB*, so as to pass through the point, *A*, of intersection of the weight and the normal reaction of the cylinder. In this case, then, the action, over a section *BC*, between two rigidly connected pieces consists of a force outside the section; which force may, of course, be replaced by one at any point in the section, together with an accompanying couple (see Art. 80).

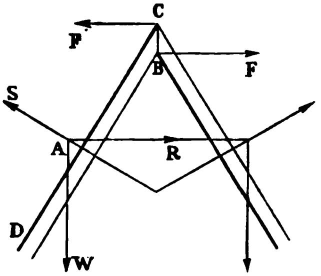

Fig. 126.

In all cases in which contact over a finite surface takes place between two bodies, the student must be careful *to examine the nature of the forces exerted between them at the individual points of contact with a view to ascertaining whether the resultant action of* *one on the other consists of a single force at all; or, if so, whether it can be assumed to act at any point in the surface of contact or must be assumed to act wholly outside it.*

107.] **Geometrico - statical Problems.** In many statical problems which relate to the positions of equilibrium of bodies the result is independent of the magnitude of some given force, and such independence can be perceived *à priori*. Thus, suppose the question to be—What is the limiting inclination to the horizon of a heavy uniform beam which rests against a rough vertical and a rough horizontal plane? In this problem we may, if we please, assume W, the weight of the beam, and $2a$, its length; but it is evident *à priori* that the result cannot involve either of these quantities. For, if the angle which the beam makes with the ground be θ, the position of equilibrium will be defined by some of the trigonometrical functions of θ, such as $\sin\theta$ or $\tan\theta$. Now, the trigonometrical functions of an angle are mere numbers, or ratios of quantities *of the same kind*. Hence, if the expression for $\tan\theta$ (suppose) involve *force*, it must involve the *ratio of one force to another force*, and if there is only *one* force given in the problem, we have no other force to combine with it in the form of a ratio or a mere number. Consequently, the weight of the beam can in no way influence its limiting inclination. Precisely similar remarks hold with regard to the only *linear* magnitude in the question, viz., the length of the beam. There is no other quantity of the same kind with which to compare it. Therefore, we are enabled to state *à priori* that the inclination of the beam to the horizon in its limiting position of equilibrium depends simply on the coefficients of friction for the beam and the two rough planes, or that

$$\theta = f(\mu, \mu'),$$

μ and μ' being these coefficients, and f denoting some (as yet) unknown function.

Again, suppose the question to be—What force applied to one of the handles of a table drawer will pull the drawer out?* It is evident that the answer must be either—no force, however great, will pull it out, or—any force, however small, will pull it out. And the result will depend simply upon the relation between the coefficient of friction for the drawer and the table, and the ratio of the side of the drawer to the distance between the handles. This is evident, because there is no given force in terms of which the required force could be expressed.

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* The friction of the bottom is neglected.

Numerous examples of this class of questions will be given in the sequel. Such problems, then, in which the result is independent of a force magnitude, we shall classify as Geometrico-statical Problems, because, though they involve principles concerning the *directions* of forces, they do not involve their *magnitudes*. In all such problems, once the requisite theorems concerning the directions of forces are made use of, the result follows at once from the geometry of the figure; and a solution by the method of resolving forces and taking moments is, in reality, an illogical process.

In connexion with the class of geometrico-statical problems, the theorem of Art. 36 will be found extremely useful.

EXAMPLES.

1. A heavy bar rests on two smooth inclined planes whose intersection is a horizontal line, the bar lying in a vertical plane perpendicular to this line of intersection; find the position of equilibrium and the pressures on the planes.

Let a and b be the segments, AG and BG, of the bar, made by its centre of gravity, G; θ the inclination of the bar to the horizon, α and β the inclinations of the planes, R and R' the pressures on these planes, respectively, and W the weight of the bar.

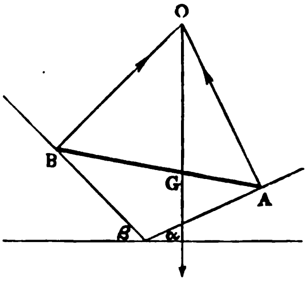

Fig. 127.

Then, since the bar is in equilibrium under the action of only three forces, they must meet in a point, O. Now the angles GOA and GOB are equal to α and β, respectively, and $BGO = \frac{\pi}{2} - \theta$. Hence

$$(a+b)\cot BGO = a\cot GOA - b\cot GOB,$$

or

$$(a+b)\tan\theta = a\cot\alpha - b\cot\beta, \qquad (1)$$

which determines the position of equilibrium.

Again, by the relations between three forces in equilibrium,

$$R = W\frac{\sin\beta}{\sin(\alpha+\beta)}, \tag{2}$$

$$R' = W\frac{\sin\alpha}{\sin(\alpha+\beta)}. \tag{3}$$

Hence, if $\frac{a}{b} = \frac{\tan\alpha}{\tan\beta}$, the bar will rest in a horizontal position.

Suppose that $a\cot\alpha - b\cot\beta$ is positive, and that $(a+b)\tan\beta < a\cot\alpha - b\cot\beta$. Then, *à fortiori* $(a+b)\tan\theta < a\cot\alpha - b\cot\beta$, since θ, the angle made with the horizon by the bar in any such position as AB, is necessarily $<\beta$.

Hence, the only position of equilibrium possible is either one of continuous contact with the plane (β), or one of continuous contact with the plane (α). Suppose the first, as in Fig. 128. To find in this case the point through which the resultant pressure of the plane (β) on the bar acts, draw AO perpendicular to the plane (α); then AO is the line of action of the pressure on this plane.

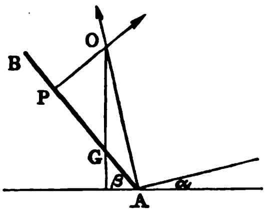

Fig. 128.

Let AO meet the vertical through G in O, and from O draw OP perpendicular to the plane (β). Evidently, P is the point at which the resultant pressure of the plane (β) acts.

But it may now be shown that, with the two inequalities supposed, this position is impossible. For if $AP > a+b$, it will be impossible; that is, if $a\frac{\cos\beta\sin(\alpha+\beta)}{\sin\alpha} > a+b$; or

$a\sin\beta\cos\beta(\cot\alpha - \tan\beta) > b$; or $a\tan\beta(\cot\alpha - \tan\beta) > b + b\tan^2\beta$; or $a(\cot\alpha - \tan\beta) > b\cot\beta + b\tan\beta$; or $a\cot\alpha - b\cot\beta > (a+b)\tan\beta$. But, by supposition $a\cot\alpha - b\cot\beta$ is positive and $>(a+b)\tan\beta$, therefore $AP > AB$, which is manifestly impossible. Hence the only position of equilibrium in this case is one of continuous contact with the plane (α). [We have supposed all through that the end A of the bar is to rest on the plane (α).] The least inclination of the plane (α) which will allow of a position of continuous contact with (β) is found by drawing at B a perpendicular to the plane (β) and joining its point of intersection with the vertical through G with A. The joining line is the normal to the plane of least inclination (α).

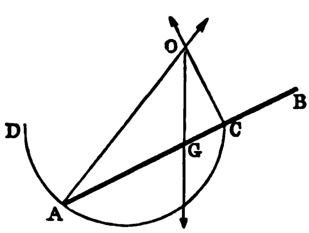

Fig. 129.

2. A uniform heavy bar, AB (Fig. 129), rests with one extremity,

A, against the internal surface of a smooth fixed hemisphere while it is supported at some point in its length by the rim of the hemisphere; find the position of equilibrium, the plane of the rim being horizontal.

It is *à priori* evident that the result must be independent of force, since the weight of the bar is the only force that may be supposed to be given; and it is also evident that the result depends on the only two linear magnitudes which may be supposed to be given—viz. the length of the bar, $2a$, and the radius, r, of the sphere.

Draw the three forces which keep the bar in equilibrium. They are the weight, a reaction at A perpendicular to the *surface of contact*, and therefore perpendicular to the sphere, and a reaction at C which for the same reason is perpendicular to the bar. These must meet in a point, O. Let $\theta =$ the inclination of the bar to the horizon $= \angle ACD$. Let the line OG meet the semicircle DAC in the point Q. Then AQ is a horizontal line. Also $\angle QAG = \angle DCA = \theta$, therefore $\angle OAQ = 2\theta$. Hence $AQ = AO\cos 2\theta$, and also $AQ = AG\cos\theta$; therefore $2r\cos 2\theta = a\cos\theta$,

or $$4r\cos^2\theta - a\cos\theta - 2r = 0.$$

This equation gives two values of $\cos\theta$, one of which supposes the hemisphere to be completed into a sphere, the end A of the bar to rest against the *upper* portion of the sphere, and the action of the sphere on A to consist of a *pull*. The student will have no difficulty in representing this position, or in proving that the reaction at $C = W\dfrac{a}{2r}$.

3. Find the position of equilibrium of a uniform heavy bar, one end of which rests against a smooth vertical plane, and the other against the internal surface of a given fixed smooth sphere.

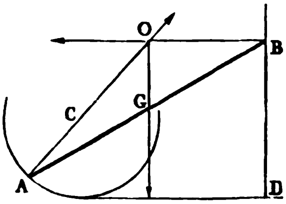

Fig. 130.

Let the length of the bar, $AB, = 2a$, $r =$ the radius of the sphere, $c =$ the distance of the centre, C, of the sphere from the vertical wall, DB; also let $\theta =$ the required inclination of the bar to the horizon, and $\phi =$ the inclination of the radius CA to the horizon.

The statics of the problem is exhausted in drawing the figure so that the weight of the bar and the two reactions at A and B shall meet in a point, O. Geometry then gives

$$2\cot OGB = \cot AOG - \cot GOB = \cot AOG,$$

or $$2\tan\theta = \tan\phi. \qquad (1)$$

Again, the perpendicular distance between A and DB is $2a\cos\theta$; but it is also evidently equal to the horizontal projection of CA + the distance of C from BD; that is,

$$2a\cos\theta = r\cos\phi + c. \qquad (2)$$

From (1) and (2) a value of θ can be obtained, and hence the position of equilibrium. These equations give θ and ϕ by the intersection of a right line with a quasi-magnetic curve (see p. 55). Thus: draw a right line $AB = c$; produce AB to C so that $BC = c$; round A as centre describe a circle of radius $2a$; round B describe one of radius r; draw any line perpendicular to AC cutting the first circle in P and the second in Q; as this line varies, trace the locus of the point of intersection of AP and BQ (a small portion of it will suffice); if this locus cuts the perpendicular to AC drawn at C in the point M, the angles θ and ϕ are MAC and MBC.

If the bar rest on the convex surface, the only change in the equations will be a change of the sign of r in (2).

4. The extremities of a bar rest at two given points against two given smooth curves in the same vertical plane; the bar is to be sustained by a cord attached to its centre of gravity and to a fixed point. Determine the position of this point so that the cord may be the weakest possible.

Let AB (Fig. 131) be the bar, G its centre of gravity, O the point of intersection of the normal reactions of the curves A and B; k the length of the perpendicular from O on the line of action of the weight, W, of the bar; p the perpendicular from O on the direction, GP, of the cord, and T the tension of the cord.

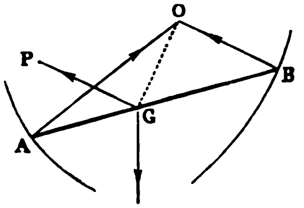

Fig. 131.

Then, taking moments about O,

$$T.p = W.k,$$

or

$$T = W\frac{k}{p}.$$

Hence, since W and k are given, T will be a minimum when p is a maximum. But the maximum value of the perpendicular from O on a right line through G is OG; hence the cord must assume a direction perpendicular to OG.

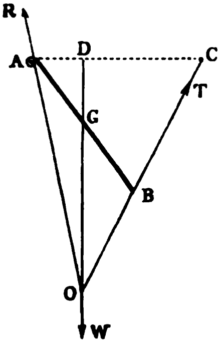

Fig. 132.

5. A heavy uniform trap-door, AB (Fig. 132), is moveable about a hinge-line represented by A; and to the middle point, B, of the opposite edge is attached a rope, BC, the extremity C of the rope being fastened to the point occupied by B when the door is horizontal. Given the length of the rope, find the magnitude and direction of the pressure on the hinge-line, and the tension of the rope.

Produce the line of the rope to meet the line of action of the weight in a point, O. Then, since the door is in equilibrium under the influence of only three forces, they must meet in a point. Hence the pressure on the hinge-line must pass through O, and since the plane of the tension, T, and the weight, W, intersects the hinge-line at A, the pressure, R, must act through A (the hinge being smooth).

To determine T, take moments about A. Then, if $p =$ the perpendicular from A on BC,

$$T \cdot p = W \cdot AD.$$

Let the angle $BAC = 2\alpha$, and let $AB = 2a$. Then $p = 2a \cos \alpha$, $AD = a \cos 2\alpha$, therefore

$$T = \tfrac{1}{2} W \frac{\cos 2\alpha}{\cos \alpha}.$$

Again, by the triangle of forces we have

$$R^2 = W^2 + T^2 - 2TW \cos \alpha;$$

and substituting the above value of T, this gives

$$R = \tfrac{1}{2} W \sqrt{4 \sin^2 \alpha + \sec^2 \alpha}.$$

The values of T and R can be at once found in terms of the lengths AB and BC. Denoting the latter by $2l$, we have $\sin \alpha = \dfrac{l}{2a}$, therefore, &c.

6. If in the last example the rope, instead of being attached to C, passes over a smooth pulley at that point, and sustains a given weight, find the position of equilibrium, and the pressure on the hinge-line.

Let P be the weight of the suspended mass, and $\theta = \angle CAB$; then the position of equilibrium is defined by the equation

$$\cos^2 \frac{\theta}{2} - \frac{P}{W} \cos \frac{\theta}{2} - \frac{1}{2} = 0, \tag{1}$$

and

$$R^2 = P^2 - 2PW \cos \frac{\theta}{2} + W^2. \tag{2}$$

Equation (1) gives two positions of equilibrium, and since it shows that one of the values of $\cos \dfrac{\theta}{2}$ is negative, one position corresponds to a value of θ greater than 180°. Such a position, of course, supposes the door capable of revolving freely about its hinge-line through four right angles.

The student will have no difficulty in representing the position of the door in this case, or in explaining why no linear magnitude enters into the equations.

7. A uniform heavy bar, AB, rests against a smooth peg, P, and against a smooth vertical wall, AD; find the position of equilibrium and the pressures on the wall and peg.

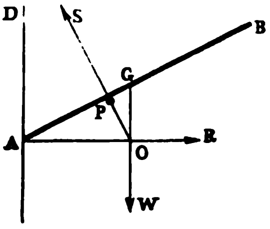

Fig. 133.

This, so far as it relates simply to the position of equilibrium, is another geometrico-statical problem. We have merely to draw AB in such a manner that the vertical through G and the perpendiculars at A and P to the wall and bar shall intersect in a common point, O.

Let $2a$ = the length of the bar, and c = the perpendicular distance of the peg from the wall. Then the position must evidently be expressed as a function of $\frac{c}{a}$. Let θ = the inclination of the bar to the vertical. Then $AP = \frac{c}{\sin\theta}$, and $AO = \frac{c}{\sin^2\theta}$. But $AO = AG.\sin\theta$; therefore

$$\frac{c}{\sin^2\theta} = a\sin\theta, \qquad \therefore \ \sin\theta = \left(\frac{c}{a}\right)^{\frac{1}{3}}. \tag{1}$$

Resolving vertically, $S.\sin\theta = W$,

$$\therefore \ S = W\left(\frac{a}{c}\right)^{\frac{1}{3}}. \tag{2}$$

Resolving horizontally, $S\cos\theta = R$,

$$\therefore \ R = W\frac{\sqrt{a^{\frac{2}{3}} - c^{\frac{2}{3}}}}{c^{\frac{1}{3}}}. \tag{3}$$

8. A triangular board, BCA (Fig. 134), of uniform thickness, rests on two smooth pegs, P and Q, at a given distance from each other, in the same horizontal line. Find its position of equilibrium.

The position of equilibrium will evidently be known if the inclination of AB to the horizon is known.

Let this inclination be θ; let the angles of the triangle be denoted by A, B, C; let $a = \angle AMC$, which the bisector, CM, of the base makes with the base; let $CM = l$, and let $PQ = k$.

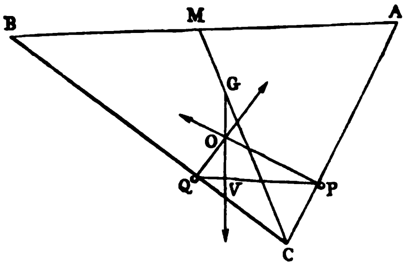

Fig. 134.

Then, since no force is given except the weight of the board, θ will depend simply on A, B, C, l, and k, and the problem is geometrical. The reactions of the pegs P and Q are perpendicular to AC and BC, respectively, and they must meet the weight of the board acting through its centre of gravity, G, in a point O. The geometry which gives the solution will express that

$$CO.\sin COV = CG.\sin CGO. \tag{1}$$

Now,

$\angle CGO = \frac{\pi}{2} + \theta - a$, and $COV = COQ - VOQ$; but $COQ = QPC$ (since the quadrilateral $QOPC$ is inscribable in a circle) $= A + \theta$; and VOQ evidently $= B - \theta$: therefore $COV = A - B + 2\theta$. Also

CO is the diameter of the circle round $QOPC$, a circle in which the chord PQ subtends at the circumference an angle $= C$;

$$\therefore \quad CO = \frac{PQ}{\sin C} = \frac{k}{\sin C}.$$

Then, since $CG = \frac{2}{3}l$, (1) becomes

$$k \sin(A - B + 2\theta) = \frac{2}{3} l \sin C . \cos(a - \theta), \tag{2}$$

an equation which determines θ, and which is easily solved graphically by the method of example 30, p. 83.

9. Two heavy uniform rods, AB and BC (Fig. 135), are connected by a smooth joint at B, and, by means of rings at A and C, are also connected with two smooth rods, AD and CD, fixed in a vertical plane. Find the reaction at the joint, the pressures at the rings, and the inclinations of the rods to the vertical in the position of equilibrium.

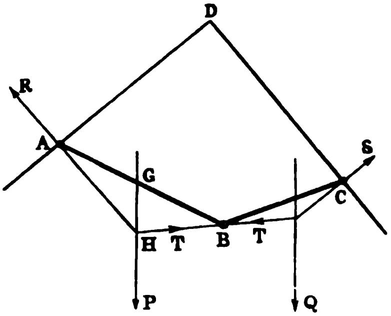

Fig. 135.

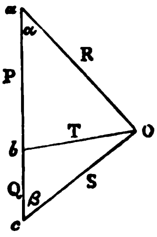

Fig. 136.

Starting from any point, O (Fig. 136), draw a force diagram of the system. Let Oa be parallel and proportional to the reaction, R, at A; let ab represent P, the weight of AB: then bO represents T, the reaction at B. In the same way let bc and cO represent Q, the weight of BC, and S the reaction at C. Let a and β be the inclinations of AD and DC to the horizon, θ and ϕ the inclinations of AB and BC to the vertical.

Then we have (from Fig. 136)

$$R = (P + Q)\frac{\sin\beta}{\sin(a+\beta)}, \tag{1}$$

$$S = (P + Q)\frac{\sin a}{\sin(a+\beta)}. \tag{2}$$

Also $T^2 = P^2 - 2PR\cos a + R^2$, which, by the substitution of the value of R from (1), becomes

$$T^2 \sin^2(a+\beta) = P^2 \sin^2 a - 2PQ \sin a \sin\beta \cos(a+\beta) + Q^2 \sin^2\beta. \tag{3}$$

Again, $\theta = HGB$, and evidently (Art. 36),

$$2\cot\theta = \cot AHG - \cot GHB$$
$$= \cot a - \cot abO \text{ (Fig. 136).}$$

Now, $\cot abO = \dfrac{P - R\cos a}{R\sin a} = \dfrac{P\cot\beta - Q\cot a}{P+Q}$, by equation (1).

Hence
$$\cot\theta = \frac{P(\cot a - \cot\beta) + 2Q\cot a}{2(P+Q)}, \tag{4}$$

and we find a similar expression for $\cot\phi$.

10. A board, $ABC,\ldots$(Fig. 137), in the shape of a regular polygon of n sides, rests at one corner, A, against a smooth vertical wall, AP, the adjacent corner, B, being attached to the wall by a string whose length is equal to the side of the polygon. Find the position of equilibrium.

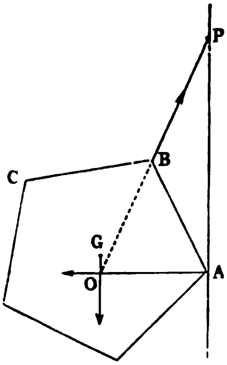

Fig. 137.

Let θ be the inclination, BAP, of the side AB to the vertical; and let O be the point in which the lines of action of the normal pressure at A, the weight of the board, and the tension of the string meet. Then, to determine θ, we have

$$OA = AP\tan\theta,$$

and
$$OA = AG\cos OAG = AG\sin GAP,$$
$$\therefore\ AP\tan\theta = AG\sin GAP.$$

Now, $GAP = GAB + \theta = \dfrac{\pi}{2} - \dfrac{\pi}{n} + \theta$; and if $a =$ the side AB, $AP = 2a\cos\theta$; $AG = \dfrac{a}{2\cos GAB}$; therefore

$$4\sin\theta\sin\frac{\pi}{n} = \cos\left(\frac{\pi}{n} - \theta\right),$$

or
$$\tan\theta = \frac{1}{3}\cot\frac{\pi}{n}.$$

This equation determines the position of equilibrium.

The pressure at A is evidently equal to $\dfrac{W}{3}\cot\dfrac{\pi}{n}$, W being the weight of the board.

11. A heavy plane body, ABC (Fig. 138), of any shape, is suspended from a smooth peg, fixed in a vertical wall, by means of a string of given length, the extremities of which are attached to two fixed points, F and F', in the body. Determine the positions of equilibrium.

Let the ellipse $P_1P_2P_3$ be described with foci F and F', and axis major equal to the length of the string. The peg will then be somewhere on this ellipse, suppose at P_2. Now, when the body is suspended from the peg, it is kept in equilibrium by its own weight acting vertically through the centre of gravity, and the two tensions in P_2F and P_2F'. But since the peg is smooth, these tensions are equal, and their resultant must bisect the angle FP_2F'; its line of action is, therefore, normal to the ellipse. And if G is the centre of gravity of the body, the resultant tension must pass through G, and be equal and opposite to the weight of the body. Hence the problem is solved by drawing normals from G to the ellipse, and then hanging the figure from the peg in such a manner that any one of these normals is vertical. Now, if G is inside the evolute, four normals can be drawn to the ellipse; but it is easy to see that only three are relevant to the solution if G is inside the lower half of the evolute (as in Fig. 138), or only one if G is inside the upper half. For, the tangents drawn to the lower half of the evolute belong to the upper half of the ellipse; and in order that the strings should be stretched, it is necessary that the peg should lie somewhere in the upper half of the ellipse. If GP_1, GP_2, and GP_3, are the normals drawn from G, the figure must be placed in a position in which any one of these lines is vertical.

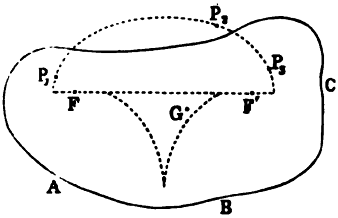

Fig. 138.

12. A rod, whose centre of gravity divides it into two segments a and b, is placed inside a smooth sphere; find the position of equilibrium.

Ans. Let θ be the inclination of the rod to the horizon, and 2α the angle subtended by the rod at the centre of the sphere; then

$$\tan\theta = \frac{a-b}{a+b}\tan\alpha.$$

13. A heavy carriage wheel is to be dragged over an obstacle on a horizontal plane by a horizontal force applied to the centre of the wheel; find the magnitude of the required force.

Ans. Let W be the weight and r the radius of the wheel, h the height of the obstacle, and F the requisite force: then

$$F = W\frac{\sqrt{2rh-h^2}}{r-h}.$$

14. If it be attempted to drag the wheel over a smooth obstacle by means of a force whose line of action does not pass through the centre, what happens? Is the result in last example modified if there is friction between the wheel and the obstacle?

15. A heavy uniform beam, AB, rests with one end, B, against a smooth inclined plane, while the other end, A, is connected with a rope which passes over a pulley and supports a given mass; find the position of equilibrium.

Ans. If α, θ, and ϕ, are the inclinations of the plane, beam, and rope to the horizon, W and P the weights of the beam and the suspended mass, respectively, the position of equilibrium is defined by the equations

$$P\cos(\phi-\alpha) = W\sin\alpha,$$

$$2\tan\theta = \tan\phi - \cot\alpha.$$

The student will easily explain why no *linear* magnitude enters into the result.

16. A rectangular board (or parallelopiped) is sustained on a smooth inclined plane by a string attached to its upper corner; the string passes over a smooth pulley and sustains a weight. Find the magnitude of this weight corresponding to a given direction of the string, and find also the pressure on the plane.

Ans. Let i be the inclination of the plane, θ the angle made by the string with the plane, W the weight of the board, P the weight of the suspended mass, and R the pressure; then

$$P = W\frac{\sin i}{\cos\theta}; \qquad R = W\frac{\cos(\theta+i)}{\cos\theta}.$$

17. Show that a rectangular board (or parallelopiped) cannot be sustained on a smooth inclined plane by a string attached to its upper corner, if the inclination of the plane is greater than the angle made by the diagonal of the board with one of the sides perpendicular to the plane.

18. A heavy uniform beam, moveable in a vertical plane about a smooth hinge fixed at one extremity, is to be sustained in a given position by means of a rope attached to the other extremity; find, geometrically, the least value of the pressure on the hinge, and the corresponding direction of the rope.

Ans. The least pressure on the hinge $= \frac{1}{2}W\sin\alpha$, W being the weight of the beam and α its inclination to the vertical. Also if θ is the angle made by the rope with the vertical when the pressure is least,

$$\cot\theta = 2\cot\alpha + \tan\alpha.$$

19. A vertical post, loosely fitted into the ground, is exposed to a uniform gale of wind; a rope of given length is to be attached to the post and to the ground; find how the attachment is to be made, in order that the rope may be least likely to break.

Ans. If h is the height of the post and if the length of the rope is $< h\sqrt{2}$, the rope must make an angle of 45° with the horizon; but if the length is $> h\sqrt{2}$, the rope must be attached to the top of the post. (See example 4.)

20. A heavy uniform bar, AB, is moveable in a vertical plane round a smooth horizontal axis fixed at A; to the end B is attached a cord which, passing over a pulley fixed at C vertically over A sustains a mass of weight, P; find the position of equilibrium.

Ans. If $AB = 2a$, $AC = b$, weight of bar $= W$, $\theta =$ inclination to the vertical,

$$\cos\theta = \frac{(4P^2 - W^2)b^2 - 4W^2a^2}{4abW^2}.$$

21. A ladder, moveable in a vertical plane about one extremity, A, is to be lowered into a vertical position by means of a rope attached to a given point, P, of the ladder, the rope passing over a pulley, C, which is vertically above A; find the tension of this rope in any position, and supposing that the rope will just sustain a tension Q without breaking, find whether or not it will break in the operation of lowering the ladder.

Ans. If W is the weight of the ladder, whose centre of gravity is distant a from A; $AC = c$; $AP = b$; $CP = r$ in any position of the ladder; T = tension of rope in this position,

$$T = W\frac{ar}{bc},$$

and the rope will break if $Q < W\dfrac{a(c+a)}{bc}$.

22. A heavy bar, AB, rests with one extremity, A, placed at the line of intersection of a smooth horizontal and a smooth inclined plane, a given point, C, in the bar being attached to a rope which, passing over a smooth pulley at a given point, D, in the inclined plane, sustains a given mass, P; find the position of equilibrium.

Ans. Let θ be the inclination of the bar, a the inclination of the plane, and ϕ the inclination of the rope, to the vertical; a the distance of the centre of gravity of bar, b the distance of the pulley, from A; $c = AC$; W= mass of bar. Then the position of equilibrium is defined by the equations

$$Wa\sin\theta = Pb\sin(\phi - a),$$
$$c\sin(\theta - \phi) = b\sin(\phi - a),$$

which may be solved graphically thus: draw a line $LM = b$; at M draw the line MN making the angle $LMN = a$; round L as centre describe a circle of radius c; round M as centre describe one of radius $\frac{Pb}{Wa}\cdot c$; take a point R on this circle such that if RL meets the first circle in Q, the perpendicular from R on MN = the perpendicular from Q on ML; then $RML = \phi$ and $RLM = \theta$.

23. If the two inclined planes were removed, a smooth horizontal axis substituted at A, round which axis the bar could turn, would the problem be in all respects the same?

24. A bar, AB, whose weight is negligible, passes through two fixed rings, C, D, which fit it tightly and presses at its end B against a fixed surface, a given force, P, being applied to the bar at its end A in the direction AB; find the pressures on the rings and on the surface at B.

Ans. If $BC = a$, $BD = b$, i = angle between AB and surface pressed, N = pressure at B, S = pressure at C, R = pressure at D,

$$N = P\operatorname{cosec} i; \qquad S = P\frac{b\cot i}{a-b}; \qquad R = P\frac{a\cot i}{a-b}.$$

25. A heavy uniform circular board is freely moveable in a

vertical plane round a horizontal axis fixed at a point O on its circumference; from two given points A and B on its circumference two masses, P and Q, respectively, are suspended; find the position of equilibrium.

Ans. If C is the centre, OCO' the diameter through O, and if $\angle ACO' = \alpha$, $\angle BCO' = \beta$, θ the required inclination of OC to the vertical, and $W =$ mass of board,

$$\tan\theta = \frac{P\sin\alpha - Q\sin\beta}{2P\cos^2\frac{\alpha}{2} + 2Q\cos^2\frac{\beta}{2} + W}.$$

26. A rectangular board, $ABCD$, of uniform thickness, is moveable in a vertical plane about a smooth hinge, P, in the side AD; the side AB is to rest, at a given inclination to the horizon, against a smooth peg, Q: find the position of this peg when the pressure on the hinge is equal to the weight of the board.

Ans. Let O be the point of meeting of the forces which keep the board in equilibrium, and G the centre of gravity of the board. Then QO must bisect the angle POG. Hence from P draw a line, PO, making the same angle with the side AB as AB makes with the vertical; and from the point, O, of intersection of this line with the vertical through G draw a perpendicular, OQ, on AB. This determines Q.

27. Find the position of the peg when the pressure on it is equal to the weight of the board, the inclination being fixed.

Ans. Let PH be the horizontal line through P meeting AB in H; produce AH to K so that $HK = HP$; then KP is the direction of pressure on hinge; therefore, &c.

28. A heavy body of any form is moveable round a smooth axis perpendicular to the vertical plane passing through the centre of gravity, and is sustained in a given position by a rope whose weight may be neglected. If the pressure on the axis bears a constant ratio to the weight of the body, prove that the direction of the rope must be a tangent to a conic whose directrix is the vertical line through the centre of gravity, and focus the point in which the axis of suspension cuts the above-mentioned vertical plane.

If, in the last example, QO be the direction of the rope, the ratio $\dfrac{\sin POQ}{\sin QOG}$ is given, and the envelope of QO, as the direction PO varies, is a conic whose focus is P, directrix GO, and eccentricity the given ratio.

29. In example 26, if the hinge is at the corner A, and the position of the peg is given, find the magnitude of the pressure on the hinge.

Ans. Let $c =$ half the length of the diagonal, $\alpha =$ angle between the diagonal and the side AB, $x =$ the distance of peg from A, $\beta =$ inclination of AB to the vertical; then the pressure on the hinge is

$$W.\frac{\sqrt{x^2 - 2cx\sin\beta\sin(\alpha+\beta) + c^2\sin^2(\alpha+\beta)}}{x}.$$

30. In the last example, find the position of the peg when the pressure on the hinge is a minimum, and the minimum value.

Ans. At the point in AB vertically under the centre of gravity on the board. The minimum pressure $= W \cos \beta$.

It is easily seen that if the hinge is anywhere along the side AD, the pressure on it will be least when the direction of this pressure is parallel to AB. [By triangle of forces.] Hence the position of the peg.

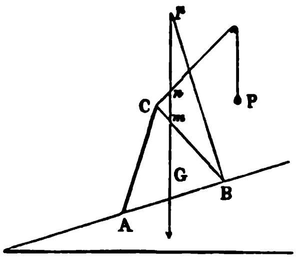

Fig. 139.

31. A rectangular board of uniform thickness rests in a vertical plane, with two of its adjacent sides in contact with two smooth pegs in the same horizontal line; find the position of equilibrium.

Ans. If P and Q (see Fig. 134) be the two pegs, CA and CB the sides in contact with P and Q, respectively, a the angle made by the diagonal CD with CB, θ the inclination of this diagonal to the horizon, c half the length of the diagonal, and l the distance PQ, the position of equilibrium is given by the equation

$$c \cos \theta = l \cos 2(\theta - a),$$

which is solved graphically by the aid of two circles, as in example 30, p. 83.

32. A triangular board (or prism), ABC (Fig. 139), of uniform thickness, is placed with its base on a smooth inclined plane, its vertex being connected with a string which passes over a smooth pulley and sustains a weight. Find the conditions of equilibrium.

Ans. Assuming the inclination of the plane to be fixed, the string must take such a direction that the perpendicular let fall on the plane from the point of intersection of the string with the vertical line, Gm, through the centre of gravity of the board, falls inside the base. Hence, if Bp be the perpendicular at the extreme point of the base, and if the string cannot cross the surface of the board, all possible directions of the string are included between Cm and Cp. Again, supposing the string to have a direction, Cn, consistent with the possibility of equilibrium, the weight P and the reaction of the plane are thus found: From n let fall a perpendicular on AB, meeting it in a point, q, suppose. Then qn is the line of action of the reaction on the plane: and, resolving along the plane, we have $W \sin i = P \cos \theta$, i being the inclination of the plane, and θ the angle which the string Cn makes with the plane. This equation determines the magnitude of P corresponding to the direction, Cn, of the string. If P is a little greater than the value thus found, the board will begin to slip up, and if P is less than this value, the board will begin to slip down the plane.

33. If in the last example the string is parallel to the plane, find the greatest inclination of the plane consistent with equilibrium.

Ans. $\text{Tan}^{-1}(\frac{1}{3} \cot A + \cot B)$.

34. If in the same example the string, instead of passing over a pulley and sustaining a weight, is knotted to a fixed peg, how are the previous conditions of equilibrium modified?

Ans. The only condition to be satisfied is that which has reference to the direction of the string. This direction must be somewhere between Cm and Cp.

35. If a rectangular picture be hung from a smooth peg by means of a string, of length $2a$, attached to two points symmetrically placed at a distance $2c$ from each other on the upper side of the frame, show that the only position of equilibrium is one in which this side is horizontal if the adjacent side of the frame is greater than $\frac{2c^2}{\sqrt{a^2-c^2}}$.

36. A rod whose centre of gravity is not its middle point is hung from a smooth peg by means of a string attached to its extremities; find the positions of equilibrium.

Ans. There are two positions in which the rod hangs vertically, and there is a third thus defined: let F be the extremity of the rod remote from the centre of gravity, k the distance of the centre of gravity from the middle point of the rod, $2a$ the length of the string, and $2c$ the length of the rod; then measure on the string a length FP from F equal to $a\left(1+\frac{k}{c}\right)$, and place the point P over the peg. This will define a third position of equilibrium.

37. A bar, BC, moveable round a smooth horizontal axis at a point A in it, has a cord attached to its upper end B, the other end of the cord being fixed at a point D in the horizontal line through A; the cord carries a small ring, capable of sliding freely along it, from which a given mass, P, is suspended; find the position of equilibrium.

Ans. Let θ = inclination of BC, and ϕ = inclination of cord at D, to the horizon; W = weight of bar whose centre of gravity is distant a from A; $BA = b$; $DA = c$; l = total length of cord. Then

$$b\cos\theta + l\cos\phi = c\,; \quad \frac{\tan\theta}{\tan\phi} = \frac{2\,Wa - Pb}{Pb},$$

which at once give θ and ϕ by the intersection of a right line with a quasi-magnetic curve (see p. 55).

38. A heavy regular polygon of any number of sides is attached to a smooth vertical wall by a string which is fastened to the middle point of one of its sides; the plane of the polygon is vertical and perpendicular to the wall, and one end of the side to which the string is attached rests against the wall. For a given position of the polygon, find the requisite direction of the string, and show that in all positions of equilibrium the tension of the string and the pressure on the wall are constant.

Ans. Let A be the vertex of the polygon in contact with the wall, G the centre of gravity, O the point in which the weight and the reaction of the wall meet, and M the middle point of the side to which the string is attached. Then the direction of the string is OM, and, the quadrilateral $GOMA$ being inscribable in a circle, the angle between the string and the vertical is constant and equal to half the angle of the polygon.

39. A square board rests with one corner against a smooth vertical wall, the adjacent corner being attached to the wall by a string whose length is equal to the side of the board; prove geometrically that the distances of the corners from the wall are proportional to 1, 3, and 4.

40. One end, A, of a heavy uniform bar rests against a smooth horizontal plane, and the other end, B, rests against a smooth inclined plane; a cord attached to B passes over a smooth pulley situated in the inclined plane, and sustains a given mass; find the position of equilibrium.

Let θ be the inclination of the bar to the horizon, a the inclination of the inclined plane, W the weight of the bar, and P the weight of the suspended mass; then the position of equilibrium is defined by the equation

$$\cos\theta\,(W\sin a - 2P) = 0. \tag{1}$$

Hence we draw two conclusions:—

(*a*) If the given quantities satisfy the equation $W\sin a - 2P = 0$, the bar will rest in all positions.

(*b*) There is one position of equilibrium, namely, that in which the bar is vertical; but this position will be impossible unless

$$P > W\sin a.$$

This position requires that both planes be conceived as prolonged through their line of intersection.

41. A uniform beam, AB, moveable in a vertical plane about a smooth horizontal axis fixed at one extremity, A, is attached by means of a rope BC, whose weight is negligible, to a fixed point, C, in the horizontal line through A; show that as the point C varies, the position of the beam being always the same, the magnitudes and lines of action of the pressure on the axis will be represented by lines drawn from A to a certain right line parallel to AB; and if the position of the beam varies, while AC is always equal to AB, find the curve whose radii vectores will represent the pressure on the axis.

42. A thin hemispherical bowl of uniform thickness rests on a horizontal plane, and a uniform bar rests partly inside and partly outside the bowl, being entirely supported by it; find the position of equilibrium. (College Examination, Cambridge, 1887.)

Ans. If $W =$ weight of bowl, $w =$ weight of bar, $r =$ radius of bowl, $2a =$ length of bar, $\theta =$ inclination of bar, and $\phi =$ inclination of the plane of the rim of the bowl to the horizon,

$$a\cos\theta = 2r\cos(2\theta+\phi) = \frac{W}{w}r\sin\phi, \tag{a}$$

which give the following construction for θ: on $AB\ (=2a)$ as diameter describe a circle, centre C; on the diameter, CD, perpendicular to AB take D such that $CD = \frac{w}{W}\,.\,a$; then if P is the point on the circle such that $DP.\cos\theta = 2r$, where $\theta = \angle PAB$, this is the required angle. It is obvious that

$$AP.DP = 4ar,$$

which shows that P is a point of intersection of the circle with a Cassinian Oval whose foci are A and D, its equation referred to them being

$$\rho\rho' = 4ar.$$

Obtain by the aid of circles alone a graphic representation of both the unknown angles in equations (a).

CHAPTER VII.

THE EQUILIBRIUM OF SYSTEMS DEDUCED FROM THE PRINCIPLE OF VIRTUAL WORK. [COPLANAR FORCES.]

108.] **Theorem.** If a particle in equilibrium under the action of any forces be constrained to maintain a fixed distance from a given fixed point, the force due to the constraint (if any) is directed towards the fixed point.

Let B be the particle, and A the fixed point. Then the string or rigid rod which connects B with A may be removed if we enclose the particle in a smooth circular tube whose centre is A; for evidently the preservation of the constancy of the distance AB receives sufficient expression in this manner. Now, in order that B may be in equilibrium inside the tube, it is necessary that the resultant of the forces acting upon it should be normal to the tube, i. e., directed towards A.

COR. 1. If A and B be two particles in equilibrium, connected by a rigid rod whose weight is neglected, the reactions of A and B on the rod are two forces equal in magnitude and opposite in direction.

COR. 2. If any body be in equilibrium under the action of two forces only, these forces must be equal and opposite in the same right line.

COR. 3. If a particle in equilibrium under the action of any forces is constrained to maintain a fixed distance from each of a number of other particles or points, the forces corresponding to these constraints are directed in the right lines joining the particle to each of the other particles or points.

This is evidently true whether the invariable distances are maintained by straight rigid bars or by crooked bars.

109.] **System of Particles rigidly connected.** Let there be any number of particles, m_1, m_2, m_3, ... (Fig. 140), each acted on by any forces, and connected with the others in such a way that the figure of the system is invariable.

Then, by the last Article, the force proceeding from the connection of m_1 and m_2 is in the line $m_1 m_2$, which we may imagine to be a rigid bar. Let this force be denoted by T_{12}. Similarly, let the forces in the bars $m_2 m_3$ and $m_3 m_1$ be denoted by T_{23} and T_{31} respectively. These internal forces may tend either to increase the distances between the particles or to diminish them. In the figure we have supposed the latter to be the case, but the result will be the same if the former supposition is made.

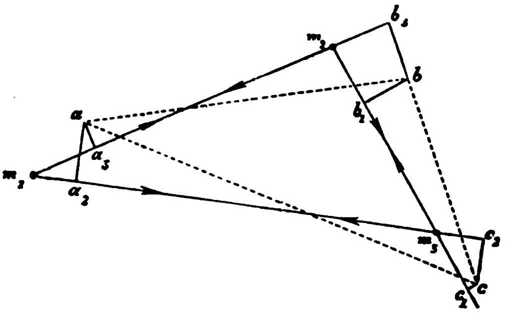

Fig. 140.

Imagine that the system is slightly displaced so as to occupy the position *abc*. Now, it has been already proved (Art. 71) that the equation of virtual work for two particles rigidly connected will not involve the force due to the connection; but, for clearness, we reproduce the proof here.

Let fall the perpendiculars aa_2 and aa_3 on the lines m_1m_3 and m_1m_2; bb_1 and bb_3, on m_2m_3 and m_1m_2; cc_1 and cc_2 on m_2m_3 and m_1m_3. Let the sum of the virtual works of the external forces (not including T_{12} and T_{13}) acting on m_1 be denoted by $\Sigma P\delta p$, and let $\Sigma Q\delta q$ and $\Sigma R\delta r$ denote similar quantities for m_2 and m_3. Then the equation of virtual work for m_1 is evidently

$$\Sigma P\delta p + T_{12} \,.\, m_1 a_3 + T_{13} \,.\, m_1 a_2 = 0 ; \tag{1}$$

that for m_2 is

$$\Sigma Q\delta q - T_{12} \,.\, m_2 b_3 + T_{23} \,.\, m_2 b_1 = 0 ; \tag{2}$$

and that for m_3 is

$$\Sigma R\delta r - T_{13} \,.\, m_3 c_2 - T_{23} \,.\, m_3 c_1 = 0. \tag{3}$$

Now (Art. 69) $m_1 a_3 = m_2 b_3$; $m_1 a_2 = m_3 c_2$; $m_2 b_1 = m_3 c_1$.

Hence, by addition, the internal forces disappear, and the equation of virtual work for the whole system is

$$\Sigma P\delta p + \Sigma Q\delta q + \Sigma R\delta r = 0,$$

or

$$\Sigma (P\delta p + Q\delta q + R\delta r) = 0. \qquad (4)$$

The same result is evidently true, whatever be the number of particles forming the system; and it is well to note that we have been enabled to obtain equation (4) connecting the external forces acting on the system, by choosing *a virtual displacement compatible with the geometrical conditions of the system*, that is, in the present case, a virtual displacement which allows the mutual distances of the particles to remain unaltered; or, again, *such a virtual displacement as might be an actual one*; for the system could *actually* occupy the position *abc*.

110.] **Elimination of the Internal Forces of a System.** By the *Internal Forces* of a system it is already sufficiently clear that we mean forces proceeding from the internal connections of the parts of the system among themselves. Such forces are directed from particle to particle, and will contribute nothing to the equation of virtual work of the system, if in the virtual displacement the distance between every two particles remains the same as before.

It is evident that if the virtual displacement violates any geometrical condition of the system, the corresponding internal force will appear in the equation of virtual work. Thus, if in Fig. 140 the distance *ab* is not equal to the distance between m_1 and m_2, we shall have by addition the term

$$T_{12} \cdot (m_1 a_3 - m_2 b_3),$$

or

$$-T_{12} \cdot \delta(m_1 m_2),$$

where $\delta(m_1 m_2)$ denotes the change or variation of the distance between m_1 and m_2.

111.] **General Equation of Virtual Work for Forces acting in one Plane on a Rigid Body*.** If the particles $m_1, m_2, m_3, \ldots$

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* We formally confine the discussion for the present to Rigid Bodies, although it is clear from last Article that what follows is applicable to systems such as freely articulated bars which, without being rigid systems, satisfy certain geometrical conditions that are not violated in the virtual displacement; and it is equally clear that these conditions may be violated if we include in our equations the work of internal forces.

form a continuous body, on which forces $P_1, P_2, P_3, \ldots$ act in one plane at different points $A_1, A_2, A_3, \ldots$ of the system (Fig. 141),

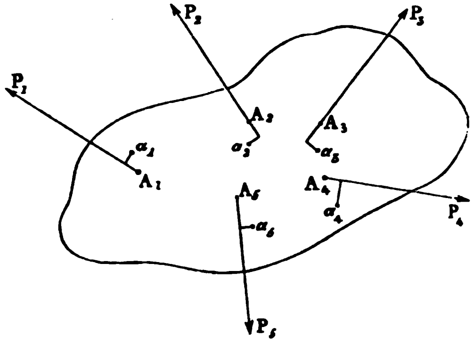

Fig. 141.

the condition necessary and sufficient for the equilibrium of the system is that the sum of the virtual works of the given forces is equal to zero for every virtual displacement which violates none of the geometrical conditions of the system.

For we have seen (Art. 67) that the condition necessary and sufficient for the equilibrium of any one particle of the system is the vanishing of the virtual work of all the forces acting upon it, the internal forces proceeding from the connection with the other particles of the system being, of course, included, as in equations (1), (2), (3) of Art. 109. Expressing thus the conditions for the equilibrium of all particles of the system, and adding the results, there remains for the condition of equilibrium the equation

$$P_1 \delta p_1 + P_2 \delta p_2 + P_3 \delta p_3 + \ldots = 0, \tag{1}$$

into which no internal force enters.

Conversely, if the sum of the virtual works of the forces vanishes for every virtual displacement, the system is in equilibrium.

For, if it is not, it will take a *determinate* motion, each point of the system describing a certain line in virtue of its connections with the other points. Now, this motion will be in no way interfered with if we introduce new connections which render it the only motion possible for the system. Under the new circumstances it is clear that if we prevent the motion of any one point, we prevent the motion of the system. Suppose the motion of the point A to be stopped by the application of a force, F, in the direction $A'A$, A' being the point to which A moves. Now, equilibrium exists under the action of (α) the given external forces, (β) the newly-introduced geometrical connections, and (γ) the force F; hence the sum of the virtual works of these forces $= 0$ for every displacement. Choose that displacement which the system is supposed actually to undergo when the force F is not applied at A. Now, by the last Article, since none of the geometrical conditions (β) are violated by this displacement, the forces proceeding from them will do no work. Hence the equation of work is

$$\Sigma P\delta p - F\,.\,AA' = 0,$$

where $\Sigma P\delta p$ denotes the virtual work of the given acting forces. But, by hypothesis, $\Sigma P\,\delta p = 0$ for every displacement, and therefore for this one; hence $F\,.\,AA' = 0$, i. e. either $AA' = 0$, or $F = 0$, either of which signifies that no motion of the system takes place. Hence the system is in equilibrium.

In Fig. 141, a_1, a_2, a_3, ... are supposed to be virtual positions of the points of application of the forces P_1, P_2, P_3 ....

112.] **Remarks on the Equation of Virtual Work.** Equation (1) of last Article, though *strictly* true in the case of forces acting on a particle, is not so when these forces are applied at points in a body of finite extension, or to a system of particles connected in any manner. In fact, the internal forces of the system have been eliminated from equations (1), (2), and (3) of Art. 109, by assuming that $m_1 a_3 - m_2 b_3 = 0$. Now, we know that this quantity is not strictly equal to zero, but equal to an infinitesimal of the second order, if the angular displacement of the line $m_1 m_2$ is regarded as an infinitesimal of the first order. It is more correct, therefore, to say that for the equilibrium of a body *the virtual work of the applied forces is an infinitesimal of the second order, if the greatest displacement in the system is regarded as an infinitesimal of the first order.*

113.] **General Uniplanar Displacement of a Rigid Body.** Since the general condition of equilibrium of a rigid body requires the vanishing of the virtual work of the acting forces for *every* virtual displacement which could be an actual one, it is evidently necessary to investigate all the kinds of displacement which such a body could undergo. Now, evidently, the position of a right line is known, if the positions of any two of its points are known; and also the position of any body is known, if the positions of any three* of its points which are not *in directum* are known. Hence, to investigate the displacements to which a rigid body may be subject, it is sufficient to determine the general displacements of a system formed of three points.

In Fig. 140 let such a system be $m_1 m_2 m_3$, and let *abc* be any displacement whatever of this system in its own plane. Then it is clear that if we moved m_1 into the position *a*, and then got m_2 into the position *b*, the remaining point, m_3, would take up the position *c*. This follows from Prop. VII of the first book of Euclid. Now what is necessary to move the line $m_1 m_2$ into the position *ab*? Two things are necessary—

(α) The point m_1 must be moved up to *a*, by a simple *motion of translation;* and

(β) When this is done, the line $m_1 m_2$ must be *rotated* about *a* so as to bring m_2 into the position *b*. This second motion is called *a motion of rotation.*

If we suppose that in the first motion (α) the line $m_1 m_2$ is moved parallel to itself, while m_1 is moved to *a*, the subsequent motion of rotation which brings m_2 into the position *b* will be a small one, the position *abc* being only slightly different from $m_1 m_2 m_3$.

Hence—*If a rigid body receives any displacement parallel to a fixed plane, it may be brought from its old into its new position by* (α) *a motion of translation which has the same magnitude and direction for all its points, and* (β) *a motion of rotation which has also the same angular magnitude and sense for all its points.*

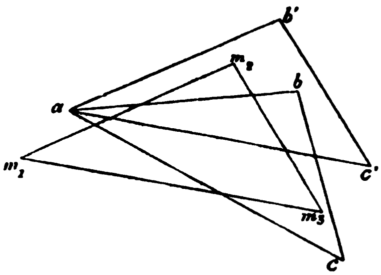

Fig. 142.

Thus, in Fig. 142, by the motion of translation common to all the points, m_1 is carried to

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* If, as in the present chapter, the displacement is made parallel to one plane, the positions of *two* points will suffice. We use the term *uniplanar* to signify 'confined to one plane.'

a, while m_2 is carried to b', and m_3 to c', the lines $m_1 m_2$, $m_2 m_3$, and $m_1 m_3$ being carried parallel to themselves to ab', $b'c'$, and ac', respectively. Then, by the motion of rotation ab' is turned round to ab, and c' is made to coincide with c.

114.] **Reduction of Displacement to Rotation.** Every uniplanar displacement of a rigid body can be produced by rotation simply. For, let $m_1 m_2 m_3$ be one position, and abc any other position, of the body, a, b, c being the displaced positions of m_1, m_2, m_3, respectively. Draw a perpendicular to the line $m_1 a$ at its middle point, and a perpendicular to $m_2 b$ at its middle point, and let these two perpendiculars intersect in I. Then $m_1 m_2 m_3$ can be brought into the position abc by a pure rotation round I. For, comparing the triangles $m_1 I m_2$ and $a I b$, we see that, since the three sides of the one are equal to the three sides of the other, the angles $m_1 I a$ and $m_2 I b$ are equal. Hence, if the body is rotated round I through the angle $m_1 I a$, so as to bring m_1 by a circular arc to a, this rotation will bring m_2 to b, and therefore every other point of the body to its proper displaced position. If the displaced position is very close to the original position, instead of bisecting $m_1 a$ and $m_2 b$ and erecting perpendiculars, we may *erect these perpendiculars at* m_1 *and* m_2 *to the directions,* $m_1 a$, $m_2 b$, *of the displacements of* m_1 *and* m_2. In this case the point I is called in Kinematics the *Instantaneous Centre* of rotation of the body.

A displacement of translation is one such that the centre of rotation is at infinity.

115.] **Virtual Work corresponding to a Virtual Motion of Translation.** Let a rigid body (Fig. 143) be in equilibrium under the action of any forces in one plane, $P_1, P_2, P_3, \ldots$, and let the body be imagined to receive a motion of translation parallel to an arbitrary line, Ox, whereby the points, A_1, A_2, A_3, ..., of application of the different forces receive virtual displacements, $A_1 a_1$, $A_2 a_2$, $A_3 a_3, \ldots$, all parallel to Ox, and equal to a. Then (Art. 57), the virtual work of the

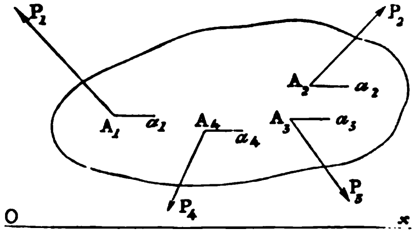

Fig. 143.

force P_1 is $a \times$ projection of P_1 along Ox. Let the projection of P_1 along Ox be X_1: then the virtual work of P_1 is aX_1. Similarly, if $X_2, X_3, \ldots$, be the components of $P_2, P_3, \ldots$ along Ox, the virtual works of these forces will be $aX_2, aX_3, \ldots$ Hence the equation of virtual work is

$$a(X_1+X_2+X_3+\ldots) = 0,$$

or

$$a\Sigma X = 0. \tag{1}$$

Consequently, since a is arbitrary, we have

$$\Sigma X = 0. \tag{2}$$

Hence—*For the equilibrium of a rigid body it is necessary that the sum of the components of the acting forces along every arbitrary right line shall be zero.*

This condition is not sufficient, since every virtual displacement of a body is not one of translation alone.

116.] **Virtual Work corresponding to a Motion of Rotation.** Let several forces, $P_1, P_2, P_3, \ldots$ (Fig. 144), act on a body at points $A_1, A_2, A_3, \ldots$, and suppose that the body is rotated through a small angle $= \omega$, round an axis perpendicular to the plane of the forces through an arbitrary point, O. Then the points $A_1, A_2, A_3, \ldots$ will describe small circular arcs, $A_1a_1, A_2a_2, A_3a_3, \ldots$ having O as their common centre, and subtending the same angle ω, at O. Let θ_1 be the angle between OA_1 and the direction of P_1. Then, evidently, the projection of A_1a_1 on the direction of P_1 is $A_1a_1 . \sin\theta_1$. But $A_1a_1 = \omega . OA_1$; therefore the virtual work of P_1 is

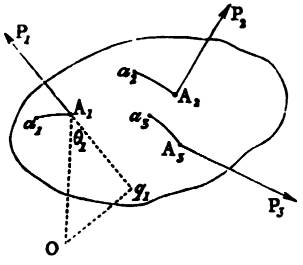

Fig. 144.

$$\omega P_1 . OA_1 \sin\theta_1.$$

If $p_1 =$ the perpendicular, Oq_1, from O on the line of action of P_1, this is evidently

$$\omega P_1 . p_1.$$

Similarly, the virtual work of P_2 is $\omega P_2 . p_2$, and that of P_3 is $-\omega P_3 . p_3$. Hence the equation of virtual work is

$$\omega(P_1p_1+P_2p_2-P_3p_3+\ldots) = 0, \tag{1}$$

or

$$\Sigma Pp = 0. \tag{2}$$

But the product of a force, P, and the perpendicular, p, let fall upon it from the point O, is the moment of the force with respect to the point O, or rather with respect to an axis through O perpendicular to the plane of the figure.

Hence, equation (2) asserts that *for equilibrium the sum (with their proper signs) of the moments of the forces with respect to any point in their plane is zero.*

As regards the signs to be given to the moments, $P_1p_1, P_2p_2, \ldots$ of the forces, we see that—

Those forces which tend to rotate the body in the same sense round the point O give virtual work of the same sign, and therefore have moments of the same sign with respect to O.

Thus, in Fig. 144, the forces P_1 and P_2 tend to turn the body round O, in a sense opposite to that of watch-hand rotation, while P_3 tends to turn it in the opposite sense. Hence, in the *Equation of Moments*, as the equation

$$\Sigma Pp = 0$$

is called, P_1p_1 and P_2p_2 have the same sign, and P_3p_3 has an opposite sign.

Since (Art. 114) every uniplanar displacement of a rigid body can be produced by a rotation, and since a rotation gives an equation of virtual work which is simply one of moments round the corresponding centre of rotation, it is clear that the necessary and sufficient conditions of equilibrium of a system of coplanar forces acting on a rigid body are exhausted in the statement—the sum of the moments of the forces round every point in their plane is zero.

Also since all possible displacements of a deformable system are by no means exhausted in motions of translation and rotation common to all its parts, the equation of virtual work for such a system does not lead to the above conditions as sufficient.

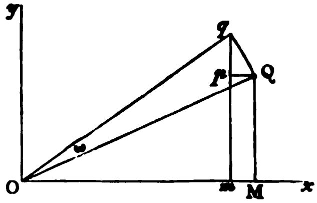

Fig. 145.

117.] Analytical Expression for the Displacement of a Rigid Body. We shall now investigate the changes produced in the co-ordinates of any point in a rigid body by given *small* motions of translation and rotation.

Let the motion of translation first take place. Then draw any two rectangular axes, Ox and Oy, through O (Fig. 145) the new position of a point O_1. Let the motion of translation O_1O, common to all parts of the body, be resolved in two components, a and b, parallel to Ox and Oy.

Then, if x and y denote the co-ordinates of a point Q_1 in the body with reference to fixed axes drawn through O_1 parallel to Ox and Oy, these quantities will be increased by a and b, respectively, by the motion of translation. To find how much they will be subsequently altered by an angular rotation $= \omega$ round O, let Q describe a small arc of a circle, Qq, round O.

Let fall the perpendiculars QM and qm on Ox, and Qp on qm. It is evident that $OM = x$ and $QM = y$. Then the increase of y produced by the rotation $= qp$, and the increase in $x = -Qp$. Now

$$Qp = Qq . \sin QOx = \omega . OQ . \sin QOx = \omega . QM = \omega y;$$

and $qp = Qq . \cos QOx = \omega . OQ . \cos QOx = \omega . OM = \omega x.$

Hence, if δx and δy denote the changes produced in x and y by the two motions combined,

$$\delta x = a - \omega y, \tag{1}$$

$$\delta y = b + \omega x. \tag{2}$$

These are the general analytical expressions for the displacements of a particle in the body. (They can obviously be obtained by differentiating the equations $x = r \cos \theta$, $y = r \sin \theta$, on the supposition that θ alone varies by a quantity $\delta\theta = \omega$, and then adding a and b to the results.)

118.] **Analytical Conditions of Equilibrium.** If any forces, $P_1, P_2, P_3, \ldots$, act on a rigid body in one plane, the condition necessary and sufficient for equilibrium is (Art. 111)

$$P_1 \delta p_1 + P_2 \delta p_2 + P_3 \delta p_3 + \ldots = 0. \tag{1}$$

Let X_1 and Y_1 be components of P_1 along two rectangular axes, Ox and Oy, and let x_1 and y_1 be the co-ordinates of the point at which P_1 acts. Then (Art. 58)

$$P_1 \delta p_1 = X_1 \delta x_1 + Y_1 \delta y_1. \tag{2}$$

Making similar substitutions for $P_2 \delta p_2$, $P_3 \delta p_3, \ldots$, equation (1) becomes

$$X_1 \delta x_1 + Y_1 \delta y_1 + X_2 \delta x_2 + Y_2 \delta y_2 + \ldots = 0, \tag{3}$$

or

$$\Sigma (X \delta x + Y \delta y) = 0. \tag{4}$$

Substituting in (4) the values of δx and δy given in the last Article, we have

$$\Sigma\,\{X(a-\omega y)+Y(b+\omega x)\} = 0,$$

or

$$a\,.\,\Sigma X+b\,.\,\Sigma Y+\omega\,.\,\Sigma(xY-yX) = 0,\ldots \tag{5}$$

since a, b, and ω are common to all points of the body, and may be taken outside the sign of summation.

Now the displacements a, b, and ω are completely independent of each other, and therefore equation (5) requires that

$$\left.\begin{array}{c}\Sigma X = 0, \quad \Sigma Y = 0\\ \Sigma(xY-yX) = 0\end{array}\right\}. \tag{6}$$

For, choose another virtual displacement in which a and b are the same as before and ω different. Then we have

$$a\Sigma X+b\Sigma Y+\omega'\Sigma(xY-yX) = 0. \tag{7}$$

Subtracting (7) from (5),

$$(\omega-\omega')\,\Sigma\,(xY-yX) = 0.$$

But since $\omega-\omega'$ is not $= 0$, this equation requires that

$$\Sigma(xY-yX) = 0.$$

Similarly, by making a alone variable, we prove that $\Sigma X = 0$, and by making b alone variable, $\Sigma Y = 0$.

The three equations (6) constitute the *analytical conditions* of equilibrium of the body.

119.] Varignon's Theorem of Moments. *The moment of the resultant of two forces with respect to any point in their plane is equal to the sum of the moments of the forces with respect to this point.* The following is the proof of this proposition by the principle of Virtual Work.

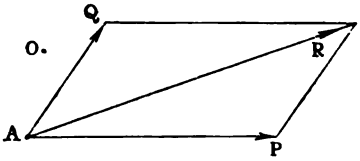

Fig. 146.

Let R (Fig. 146) be the resultant of two forces, P and Q, applied at a point A, and let O be any point in their plane. Then the virtual work of R for any displacement of A = the virtual work of P + the virtual work of Q. Let the virtual displacement of A be one of rotation round O, through a small angle $= \omega$. Then, as in Art. 116, the virtual work of R is $\omega\,.\,R\,.\,OA\,.\,\sin OAR$; but this $= \omega\,.\,R \times$ the perpendicular from O on $R = \omega \times$ the moment of R with respect to O. Similarly, the virtual work of $P = \omega \times$ moment of P with respect to O; and virtual work of $Q = \omega \times$ moment of Q with respect to O. Therefore, &c.—Q.E.D.

In precisely the same way, the moment of the resultant of any number of forces is proved to be equal to the sum of the moments of the forces separately.

120.] **Two Parallel Forces.** *To find the resultant of two parallel forces, P and Q, acting in the same sense.*

Let AB (Fig. 147) be the shortest distance between P and Q, and let the forces be supposed to act at A and B. Also let the reversed resultant, R, act at any point, O, in AB. Since the forces are in equilibrium, their virtual work $= 0$ for every virtual displacement (Art. 111). Choose first a virtual displacement of translation along AB. For this displacement the virtual work of the forces P and $Q = 0$, therefore the virtual work of $R = 0$, therefore R is parallel to P and Q. Again, choose a virtual displacement of rotation about O through an angle $= \omega$. The virtual work of P is then $P.\omega\, OA$, and that of Q is $-Q.\omega\, OB$, while that of R is zero. Hence

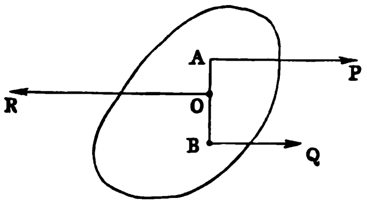

Fig. 147.

$$P.OA - Q.OB = 0, \qquad (1)$$

$$\therefore \frac{OA}{OB} = \frac{Q}{P}.$$

Finally, to find the magnitude of R, take a virtual displacement of translation parallel to the forces. This evidently gives

$$R = P + Q. \qquad (2)$$

Therefore *the resultant of two parallel forces acting in the same sense is a force parallel to them in the same sense, equal to their sum, and dividing the line joining their points of application in the inverse ratio of the forces.*

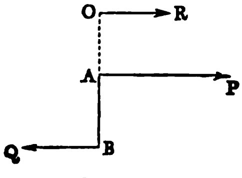

Fig. 148.

Equation (1) asserts that the moments of two parallel forces with respect to any point on their resultant are equal and opposite.

If P and Q act in opposite senses (Fig. 148), the resultant is obtained in magnitude and direction by simply changing the sign of Q.

Thus (1) becomes
$$\frac{OA}{OB} = \frac{Q}{P}, \tag{3}$$
which shows that O is on the production of AB at the side of the greater force; and (2) gives
$$R = P - Q. \tag{4}$$

EXAMPLES.

1. To solve example 9, p. 157, by the principle of Virtual Work.

Imagine a displacement in which the ends A and B remain in contact with the planes. Then the virtual works of R and S are zero, and if y is the height of G above the horizontal plane, the equation of virtual work is
$$-Wdy - P . d(AC) = 0. \tag{1}$$
Now $y = a \sin\theta$, $AC = (a+b)\cos\theta$; $\therefore\ dy = a\cos\theta d\theta$, and
$$d(AC) = -(a+b)\sin\theta d\theta;$$
$\therefore$ (1) gives $\quad Wa\cos\theta = P(a+b)\sin\theta$, or $\tan\theta = \dfrac{Wa}{P(a+b)}$.

2. To solve example 10, p. 157, by the principle of Virtual Work.

Choosing a virtual displacement which keeps A and B in contact with the planes, the equation of work is
$$-Wdy - T . d(AC) = 0. \tag{1}$$
Now $PC^2 = BP^2\cos^2\theta + AP^2\sin^2\theta$, and this equation also holds in the displaced position. Hence we may differentiate it, and we then obtain
$$\begin{aligned} PC . d(PC) &= -(PB^2 - PA^2)\sin\theta\cos\theta d\theta \\ &= -(a+b)(PB - PA)\sin\theta\cos\theta d\theta; \\ \therefore\quad d(PC) &= -(a+b)\left(\frac{\cos\phi}{\cos\theta} - \frac{\sin\phi}{\sin\theta}\right)\sin\theta\cos\theta d\theta \\ &= -(a+b)\sin(\theta - \phi)d\theta. \end{aligned}$$
Also $y = a\sin\theta$, $\therefore\ dy = a\cos\theta d\theta$; and substituting these values of $d(PC)$ and dy in (1), we obtain the value of T.

3. Four rigid bars (Fig. 149), freely jointed together at their extremities, form a quadrilateral, $ABCD$; the opposite vertices are connected by strings, AC and BD, in a state of tension; compare the tensions of these strings.

Let the bar AB be considered as fixed, and let the quadrilateral undergo any slight deformation. Then the bars AD and BC will turn round the points A and B, that is, the points D and C will describe small paths, Dd and Cc, perpendicular to AD and BC.

Hence (Art. 114) the point, I, of intersection of AD and BC is the *instantaneous centre* for the bar CD, and the angles DId and CIc are equal. Denote their common value by $\delta\theta$. Then $Dd = ID \,.\, \delta\theta$, and $Cc = IC \,.\, \delta\theta$.

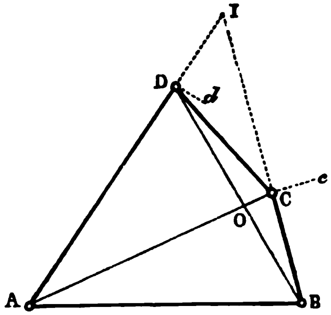

Fig. 149.

Now, since in the displacement of the system none of the geometrical conditions—namely, the constancy of the lengths of the bars —are violated, the reactions of the bars will not enter into the equation of virtual work. Hence if T and T' denote the tensions of the strings AC and BD, this equation will be (see p. 97)

$$T \,.\, \delta AC + T' \,.\, \delta BD = 0. \qquad (1)$$

But δAC = projection of Cc on AC

$$= Cc \,.\, \sin ACB = IC \,.\, \sin ACB \,.\, \delta\theta;$$

and similarly $\delta BD = -ID \,.\, \sin BDA \,.\, \delta\theta$. Hence (1) becomes

$$T \,.\, IC \,.\, \sin ACB = T' \,.\, ID \,.\, \sin BDA. \qquad (2)$$

Again, $$\frac{IC}{ID} = \frac{AC \sin CAD}{BD \sin CBD}.$$

Substituting in (2), we obtain

$$T\frac{AC}{OA \,.\, OC} = T'\frac{BD}{OB \,.\, OD}.$$

Another solution of this problem (quoted from Euler) will be found in Walton's *Mechanical Problems*, p. 101.

4. Four rigid bars, freely jointed at their extremities, form a quadrilateral, $ABCD$; the bars AB and AD are connected by a string $a\alpha$ in a state of tension, a being a given point in AB, and α a given point in AD; in the same way, BA and BC are connected by a string $b\beta$; CB and CD are connected by a string $c\gamma$; and DC and DA by a string $d\delta$; find the relation between the tensions of these strings.

If the lengths of the strings $a\alpha$, $b\beta$, $c\gamma$ and $d\delta$ are denoted by x, y, z, and w, and the tensions in them by X, Y, Z, W, the equation of virtual work for a slight deformation will be

$$X\delta x + Y\delta y + Z\delta z + W\delta w = 0. \qquad (1)$$

Now $$x^2 = Aa^2 + A\alpha^2 - 2Aa \,.\, A\alpha \cos A = Aa^2 + A\alpha^2 - 2\frac{Aa \,.\, A\alpha}{AB \,.\, AD}(AB^2 + AD^2 - BD^2);$$

therefore $$x\delta x = 2\frac{Aa \,.\, A\alpha}{AB \,.\, AD} \,.\, BD \,.\, \delta BD.$$

Substituting this value of δx, and similar values of δy, δx, δw, in (1), we have

$$\left(\frac{X}{x}\cdot\frac{Aa.A\alpha}{AB.AD}+\frac{Z}{z}\cdot\frac{Cc.C\gamma}{CB.CD}\right)BD.\delta BD$$

$$+\left(\frac{Y}{y}\cdot\frac{Bb.B\beta}{BA.BC}+\frac{W}{w}\cdot\frac{Dd.D\delta}{DC.DA}\right)AC.\delta AC=0.$$

But from the last Example, we have

$$\frac{\delta BD}{\delta AC}=-\frac{BD.OA.OC}{AC.OB.OD};$$

hence, finally, $\left(\frac{X}{x}\cdot\frac{Aa.A\alpha}{AB.AD}+\frac{Z}{z}\cdot\frac{Cc.C\gamma}{CB.CD}\right)\frac{BD^2}{OB.OD}$

$$=\left(\frac{Y}{y}\cdot\frac{Bb.B\beta}{BA.BC}+\frac{W}{w}\cdot\frac{Dd.D\delta}{DC.DA}\right)\frac{AC^2}{OA.OC}.$$

For a different solution, see Walton, *ibid.*

5. Six equal heavy bars are freely jointed at their extremities; one is fixed on a horizontal plane, and the system lies in a vertical plane; the middle points of the two upper non-horizontal bars are connected by a rope in a state of tension. Show that the tension of this rope is

$$6\,W\cot\theta,$$

W being the weight of each bar, and θ the inclination of the non-horizontal bars to the horizon.

Let x be the length of the rope, y the height of the centre of gravity of the system, $2a$ the length of each bar, and T the tension of the rope. Then the virtual work of the tension is $-T\delta x$ (see p. 181), and the virtual work of the weight of the system is $-6\,W\delta y$. Hence

$$T\delta x+6\,W\delta y=0.$$

But $x=2a(1+\cos\theta)$, and $y=2a\sin\theta$, and the deformation imagined is one in which the upper horizontal bar moves vertically through a small space. Hence the values of y and x will be of the same forms as before, and

$$\delta x=-2a\sin\theta\,\delta\theta,\quad \delta y=2a\cos\theta\,\delta\theta.$$

Substituting these values of δx and δy, we have

$$T=6\,W\cot\theta.$$

6. A body receives a small general displacement parallel to one plane; find the co-ordinates of the instantaneous centre.

If the components of the motion of translation parallel to the axes of x and y are δa and δb, and the rotation is $\delta\omega$, the equations of Art. 117 give for the displacement of any point whose co-ordinates are x, y,

$$\delta x=\delta a-y\,\delta\omega$$

$$\delta y=\delta b+x\,\delta\omega.$$

Now, the displacement of the instantaneous centre is zero; hence, if (x, y) be its co-ordinates, we have

$$x = -\frac{\delta b}{\delta \omega}, \quad y = \frac{\delta a}{\delta \omega}.$$

A particular case may be noticed. If any body in contact with a surface receives any small displacement parallel to one plane, the body still remaining in contact with the surface, the instantaneous centre lies on the normal to the surface of contact. In the pure rolling of one figure on another the point of contact is the instantaneous centre.

7. A uniform bar, AB (Fig. 133, p. 167), rests as a tangent at a point P against a smooth curve in a vertical plane, one extremity, A, resting against a smooth vertical plane; find the position of equilibrium, and the nature of the curve so that the bar may rest in all positions.

Let the weight of the bar through G, and the normal reactions at A and P meet in the point O; take the vertical line AD as axis of y; and let $2a =$ the length of the bar. Then, if x is the abscissa of P, we have $AO = \frac{x}{\sin^2\theta}$, and also $AO = a \sin\theta$. Hence, equating these values,

$$x = a \sin^3\theta. \tag{1}$$

Now, from the equation of the given curve, θ is known in terms of x in the form

$$\theta = f(x). \tag{2}$$

From (1) and (2) the value of x, and therefore the position of equilibrium, can be found.

For example, if the curve be a circle of radius r whose centre is at a distance c from the vertical plane, we find

$$a \sin^3\theta + r\cos\theta - c = 0.$$

If $r = 0$, we get the result in Ex. 7, p. 167.

If (1) holds in all positions in which the bar is placed, every position is one of equilibrium. Now, since $\tan\theta = \frac{dx}{dy}$, (1) gives

$$dy = \sqrt{a^{\frac{2}{3}} - x^{\frac{2}{3}}} \,.\, x^{-\frac{1}{3}}\, dx,$$

and since this equation holds in all positions, we may integrate it.

Hence $$y + k = -(a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}},$$

or $$x^{\frac{2}{3}} + (y+k)^{\frac{2}{3}} = a^{\frac{2}{3}},$$

k being an arbitrary constant.

We may, without loss of generality, assume $k = 0$, and the curve will be

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

The equation of virtual work shows that in this case the centre of gravity of the bar is at a constant height. For if $\bar{y}$ denote the ordinate of G, this equation is

$$W d\bar{y} = 0,$$

and since this holds in all positions, we have, by integration, $\bar{y} =$ constant.

8. Four rigid bars freely jointed at their extremities form a quadrilateral $ABCD$ (Fig. 150); the middle points of the opposite pairs of bars are connected by strings, mm' and nn', in a state of tension. Compare the tensions of these strings.

Let l and l' be the lengths of the strings mm' and nn', and let the tensions in them be T and T', respectively.

Then, assuming the quadrilateral to receive any small deformation, the equation of work will be

$$T\delta l + T'\delta l' = 0. \qquad (1)$$

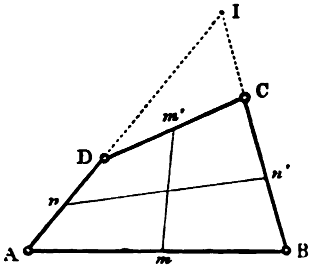

Fig. 150.

Now, it may be left to the student as an exercise to prove that

$$l'^2 - l^2 = \tfrac{1}{4}(AB^2 + CD^2 - BC^2 - AD^2),$$

that is, $l'^2 - l^2$ is constant, however the quadrilateral may be deformed. Hence

$$l\delta l - l'\delta l' = 0; \qquad (2)$$

and from (1) and (2) we have

$$\frac{T}{l} + \frac{T'}{l'} = 0, \qquad (3)$$

a remarkable result, since it shows that one of the tensions must be *negative*; i.e. if the bars AB and CD are pulled together, equilibrium will be impossible unless the bars AD and BC are pulled asunder.

It is well to notice an apparent exception to the result (3). The student will easily prove that if the sides AB and DC are parallel, equilibrium will be maintained by the single string mm' in any state of tension, i.e. $T' = 0$, a result which contradicts (3).

The difficulty is easily removed, however, by reverting to (1), which in the case under consideration is identically satisfied. For, since AB and CD are parallel, the line mm' passes through I, the instantaneous centre for CD, and therefore for a slight deformation the point m' moves perpendicularly to Im', that is, to mm'. Hence $\delta l = 0$, and equation (1) is satisfied by having at once $T' = 0$ and $\delta l = 0$. The combination of (1) and (2) is therefore irrelevant.

9. A number of bars are freely jointed together at their extremities and form a polygon; each bar is acted on perpendicularly by a force proportional to its length; all the forces emanate from one point and all act inwards or all outwards; prove by the principle of virtual work that, for equilibrium, the polygon must be inscribable in a circle.

Let the polygon be $ADCBEF..$ (Fig. 150), of which the vertices E, F,... are not represented in the figure. [AB is not one of the bars.]

Choose a virtual displacement in which all the bars except the three AD, DC, CB remain fixed, and let the extremities A and B be fixed in the displacement. Then I is the instantaneous centre for DC. Let O be the point from which the forces emanate; let m, n, p be the feet of perpendiculars from O on AD, DC, CB, respectively; let Q be the foot of the perpendicular from I on DC; let IQ meet mO in L and pO in M; and let the forces in Om, On, Op be $k.AD$, $k.DC$, $k.CB$.

If AD turns round A through the small angle $\delta\phi$, the displacement of D is $AD.\delta\phi$; and if DC turns round I through $\delta\omega$, the displacement of D is $ID.\delta\omega$. Hence

$$AD.\delta\phi = ID.\delta\omega.$$

Similarly $$BC.\delta\theta = IC.\delta\omega,$$

if $\delta\theta$ is the angle through which BC turns round B.

Now the equation of virtual work is

$$k.AD.Am.\delta\phi + k.DC.In.\delta\omega.\cos InQ - k.BC.Bp.\delta\theta = 0;$$

or, by the first two equations,

$$Am.ID + DC.nQ - Bp.IC = 0. \tag{1}$$

Now $$Im.ID = LI.IQ, \text{ and } Ip.IC = IQ.IM;$$

therefore $$Im.ID - Ip.IC = LM.IQ. \tag{2}$$

Adding (2) to (1), we have

$$AI.ID - BI.IC = LM.IQ - DC.nQ.$$

But the right side of this equation is zero, since the triangles DCI and LMO are similar (nQ is the altitude of the latter). Hence the quadrilateral $ADCB$ is inscribable in a circle; and in this circle lie also the quadrilaterals $DCBE$, $CBEF$,... and therefore the whole polygon.

10. Six equal heavy bars are freely jointed at their extremities; one bar is fixed in a horizontal position, and the system hangs in a vertical plane; the middle points of each pair of adjacent non-horizontal bars are connected by two strings in a state of tension. Show by the principle of work that, if the hexagon is regular in its position of equilibrium, the tension of each string is three times the weight of a bar.

11. Four bars whose weights may be neglected are jointed together by smooth pins and form a quadrilateral, $ABCD$, in a vertical plane. The joint A is fixed, while the lateral joints B and D rest each against a smooth fixed vertical plane. A given vertical force being applied at the point C, find the magnitudes of the reactions of the planes at B and D, and the direction and magnitude of the pressure on the joint A.

Ans. Let F be the force applied at C; P and Q the reactions at

B and *D*; *R* the force on *A*; also let α, β, γ, δ be the inclinations to the horizon of the bars *AB*, *BC*, *CD*, *DA*, and θ the angle made by *R* with the horizon. Then we shall have

$$\frac{P}{1+\cot\alpha\tan\beta}=\frac{Q}{1+\cot\delta\tan\gamma}=\frac{F}{\tan\beta+\tan\gamma};$$

$$R=\sqrt{P^2+Q^2+F^2-2PQ};$$

$$\tan\theta=\frac{\cot\beta+\cot\gamma}{\cot\alpha\cot\gamma-\cot\beta\cot\delta}.$$

(To get *P* choose a displacement of the bars in which *AD* remains fixed; the intersection of *AB* and *CD* will then be the instantaneous centre of rotation for the bar *BC*.)

12. Two heavy uniform bars, *AC* and *CB* (Fig. 165, Art. 139), are connected by a smooth joint at *C*; the bar *AC* is moveable in a vertical plane about a smooth joint fixed at *A*, and the extremity *B* of the bar *CB* is capable of moving along a smooth horizontal groove whose direction passes through *A*. It is required to keep the system in a given position by means of a horizontal force applied at *B*; determine by the principle of work the requisite magnitude of this force.

Ans. If α and α' denote the angles *CAB* and *CBA*; *W* and *W′* the weights of *AC* and *CB*; and *F* the required force,

$$F=\frac{W+W'}{2(\tan\alpha+\tan\alpha')}.$$

13. Four bars, freely articulated at their extremities, form a parallelogram, *ABCD*; two forces, each equal to *P*, act in opposite directions in the diagonal *AC*, and two forces, each equal to *Q*, act similarly in *BD*. Find the figure of equilibrium.

Ans. The adjacent sides of the parallelogram being *a* and *b*, the angle between them ω, we have

$$\cos\omega=\frac{a^2+b^2}{2ab}\cdot\frac{P^2-Q^2}{P^2+Q^2}.$$

14. If the forces in Example 9 are each transferred to the middle point of the bar on which it acts, prove by virtual work that the polygon must be inscribable for equilibrium.

15. Four rigid bars jointed together at their extremities form a plane quadrilateral *ABCD*; forces *P*, *Q*, *R*, *S* are so applied at the vertices *A*, *B*, *C*, *D*, respectively, as to preserve equilibrium; show that the lines of action of the forces must be such that the diagonals of the quadrilateral which they determine pass through the points of intersection of the opposite sides (*DA*, *CB*) and (*BA*, *CD*) of the given quadrilateral.

[This follows at once by the method of Example 3. The result is also thus expressed by Schell (*Theorie der Bewegung und der Kräfte.*

vol. ii. p. 74): Any two adjacent forces and the side at the ends of which they act form a triangle which is in perspective with that formed by the remaining pair and their corresponding side.]

16. Seven equal uniform heavy bars, freely jointed together at common extremities, form a regular heptagon, *ABCDEFG*, the system being suspended vertically from the point *A*, and the vertices *G* and *D* being connected by a weightless strut, as also the vertices *B* and *E*; find the pressure in each strut.

Ans. Taking a virtual displacement in which the vertices *A*, *B*, *C*, *D*, *E* all remain at rest, and *G* and *F* alone move, we find the pressure to be

$$2W \sin\frac{2\pi}{7},$$

where $W =$ weight of each bar.

[The student will see that *strings* would not do instead of struts.]

17. Two equal bars, *OA* and *OC*, are freely jointed at the fixed point *O*; four equal bars forming a lozenge, *ABCD*, are freely jointed at *A*, *B*, *C*, and *D*, and the system (called a *Peaucellier's Cell*) is held in equilibrium by two forces applied at *B* and *D*. If the force at *D* is of constant magnitude in all positions of the cell, as it suffers deformation about *O*, prove that the force at *B* will be one varying inversely as the square of the distance *OB*. (Mr. G. H. Darwin, *Proceedings of the London Math. Soc.*, April 8, 1875. See the same paper for Mr. Darwin's most ingenious mechanical description of the Equipotential Lines of any number of magnetic poles by means of Peaucellier's Cells.)

18. A regular pentagon, *ABCDE*, formed of equal heavy uniform bars jointed together, is suspended from the joint *A*, and the regular pentagonal form is maintained by a rod of negligible weight joining the middle points, *K*, *L*, of *BC* and *DE*; find the stresses at *K*, *B*, *C*. (*Tripos*, 1885.)

Ans. If $W =$ weight of each bar, the stress (compressive) in *KL* is $2W\cot\frac{\pi}{10}$; at *C* the force acting on the bar *CD* has a vertical upward component $\frac{1}{2}W$, and a horizontal tensile component $2W\operatorname{cosec}\frac{\pi}{5}$; at *B* the force acting on *AB* has a vertical downward component $\frac{3}{2}W$, and an outward horizontal component $2W\tan\frac{3\pi}{10}$.

19. Four equal uniform bars, each of weight *W*, freely jointed at their extremities and arranged in the form of the letter W, are suspended from the upper ends of the extreme bars, the middle points of the two inner bars being connected by a uniform rod of weight *P*; find the stress in this rod and the reactions at the joints.

Ans. The force acting on the connecting rod at each end has a vertical upward component $= \frac{1}{2}P$ and a horizontal tensile component $= (4W + \frac{3}{2}P)\tan\theta$, where 2θ is the angle between any two intersecting bars in the given figure; the force acting on the left-hand middle bar at its upper end is wholly horizontal and $=(\frac{5}{2}W+P)\tan\theta$ towards the left; on its lower end the force acting has a vertical upward component $=W+\frac{1}{2}P$, and a horizontal component towards the left $=\frac{1}{2}(3W+P)\tan\theta$; this latter being also the component of force acting on the extreme left-hand bar at its upper end, where there is also a vertical upward component $=2W+\frac{1}{2}P$.

20. Four equal uniform bars, each of weight W, are jointed together so as to form a rhombus; this is suspended vertically from one of the joints, and a sphere of weight P is balanced inside the rhombus so as to keep it from collapsing; find the figure of equilibrium.

Ans. If r is the radius of the sphere, $2a$ = length of each bar, 2θ = angle at the fixed joint,

$$\frac{\sin^3\theta}{\cos\theta}=\frac{Pr}{4(P+2W)a}.$$

121.] **Stable and Unstable Equilibrium.** When a body, or a system of connected bodies, is in equilibrium under the action of any forces, the Principle of Virtual Work signalises a position of equilibrium of the body, or an equilibrium configuration of the system of bodies, by an equation which asserts that for all imagined small displacements of the body or system the total work of the forces is zero, or rather an infinitesimal quantity of, at most, the second order, if the greatest displacement imagined is taken as an infinitesimal of the first order—just as the Differential Calculus analogously defines a maximum or a minimum value of a function of some variable or variables. But (just again as in the latter case) the work done by the forces will, in reality, be some infinitesimal quantity of, at most, the second order; and a great deal depends on whether this quantity is a positive or a negative one. If the former, the displacements will, if actually made, grow in magnitude, i. e., the system will not go back to its state of equilibrium, but will deviate more and more from it. In this case the equilibrium is said to be *unstable*. If the latter, the system will revert to equilibrium, and the equilibrium is said to be *stable*.

As this subject is fully gone into in the second volume of this treatise, it is not proposed to discuss it here; but the very simple case in which a system is in equilibrium under the action of gravity and the reactions of smooth fixed supporting surfaces or axes may be here anticipated.

For any displacements in which the points of contact are moved simply along the fixed surfaces, the equation of Virtual Work assumes the simple form

$$-W \, . \, d\bar{y} = 0,$$

where $\bar{y}$ is the *height* of the centre of gravity, G, of the system above any fixed horizontal plane, and W the weight of the whole system.

Hence, in general, a position or configuration of equilibrium is one in which the centre of gravity of the system occupies the highest or lowest position that it can consistently with the resting of the bodies against the given surfaces.

And, moreover, since if $\bar{y}$ is a maximum, any actual slight displacement would, in reality, lower G, the work of W would, in reality, be some small *positive* quantity; then equilibrium would be unstable; and if $\bar{y}$ is a minimum, any slight displacement would raise G, so that W would do some small amount of *negative* work, and the equilibrium would be stable.

Hence for such a system, under the action of gravity, its equilibrium will be stable when its centre of gravity is placed in the lowest position, and unstable when its centre of gravity is placed in the highest position, that it can occupy consistently with the given conditions of constraint.

Unstable configurations of equilibrium are of no practical value, since they would cease to exist after the slightest disturbance. For example, a rod AB moveable round a fixed smooth horizontal axis at one end, A, would theoretically be in equilibrium if its centre of gravity were placed vertically *above* A; but, practically, such a position does not exist.

Again, if the assumption as to the nature of Saturn's Ring, that it is a continuous solid or fluid, whether homogeneous or not, uniform in figure or the reverse, should succeed in explaining its relative equilibrium, but should at the same time lead to the result that its equilibrium is unstable, the assumption is as much to be rejected as if it did not explain any equilibrium at all.

Examples.

1. Two uniform rods, AB, CD, have their extremities A and C connected by a string, and their extremities B and D connected by an equal string. The strings are placed over smooth pegs in a horizontal

plane, and the rods are horizontal; find the position of equilibrium. Is it stable or unstable? (*Cambridge Examinations.*)

Ans. Stable.

2. When several liquids of different densities are superposed, without combining, in a vessel, show that they will arrange themselves in layers diminishing in density from the bottom.

3. If a diving bell of any form, whose material is of greater specific weight than water, requires the application of a vertical *downward* force in order to sink it, show that it will reach a position of unstable equilibrium when it is lowered so much that

$$\frac{p}{p_0} = \frac{P}{W - W'},$$

where p = intensity of pressure of the air in the bell in the required position,
p_0 = intensity of pressure of the atmosphere,
W = weight of the material of the bell,
W' = weight of a volume of water equal to that of the material of the bell,
P = weight of a volume of water equal to that of the interior of the bell.

4. A uniform rod, AB, of length $2a$ and small cross-section, is moveable about a smooth horizontal axis at one end A which is at a height h above the surface of a liquid, the rod partly dipping into the liquid. If w and w' are the specific weights of the liquid and the rod, and θ the inclination of the rod to the horizon in the inclined position of equilibrium, show that

$$\sin\theta = \frac{h}{2a}\left(\frac{w}{w - w'}\right)^{\frac{1}{2}},$$

and that when this position exists it is one of stability.

(Many of the previous examples may be treated with a view to the determination of stability or instability.)

CHAPTER VIII.

SIMPLE MACHINES.

122.] **Functions of a Machine.** A machine may be defined either from a statical or from a kinematical point of view. Regarded statically, it is *any instrument by means of which we may change the direction, magnitude, and point of application of a given force*; and regarded kinematically, it is *any instrument by means of which we may change the direction and velocity of a given motion.*

In Statics it is usual to consider the points or machines to which forces equilibrating each other are applied as absolutely *motionless*; nevertheless, it appears from our definition of force (Art. 1), that a system of forces acting at a point will be in equilibrium when the point has a uniform motion in a right line. If a particle describes any curve whatever with uniform velocity, a little reflection will show that at no point of its path can there be any force in the direction of the tangent—or, in other words, the force acting on it must everywhere be normal to the path. It follows (see Art. 66), that there is no work done by this force in the passage of its point of application from any one position to any other. Extending this, we shall so far anticipate the results of Kinetics as to assume that *when the parts of any machine are each in a state of uniform motion, the forces acting on the machine are in equilibrium among themselves.*

By the extension of the equilibrium of forces to this case, we comprise both the statical and kinematical definitions of a machine in the following:—a machine is *any assemblage of different pieces whose displacements, resulting from their mode of connection, depend on each other by geometrical laws, and whose object is to transform into mechanical work the result of the action of given applied forces.* (See Resal, *Mécanique Générale*, vol. iii. p. 3.)

It has been already pointed out that in applying the equation of virtual work to a system of connected bodies, advantage is gained by choosing such displacements as do not violate any of the geometrical connections of the system. This principle we shall use largely in the discussion of machines, and the displacements which we shall choose will be those which the different parts of a machine actually undergo when it is employed in doing work. Thus, instead of equations of *virtual* work, we shall have equations of *actual* work; and in future we shall speak of the principle referred to as the *Principle of Work*.

Since in the motion of a machine the work done by a force applied to any part of it depends on the magnitude and direction of the displacement of the point of application of this force, we see at once the importance of the discussion of the motions produced in the several parts of a machine by a definite motion given to some one part. This discussion, which is a problem of pure geometry, constitutes the *Kinematics of Machinery*, for which the student may consult Resal's *Mécanique Générale*, or the treatise of Reuleaux.

123.] **Efforts and Resistances.** Every machine is designed for the purpose of overcoming certain forces which are called *resistances*; and the forces which are applied to the machine to produce this effect are called *efforts*. The distinction between these forces is easily drawn by the Principle of Work. For, when the machine is in motion, every effort displaces its point of application in its own direction, while the point of application of a resistance is displaced in a direction *opposite* to that of the resistance. An effort is, therefore, one whose elementary work is positive, and a resistance one whose elementary work is negative.

An effort applied to a machine is often (but very improperly) called a *power*. The resistances against which a machine works are divided into two classes, viz. *useful resistances* and *wasteful resistances*. The former constitute those which the machine is specially designed to overcome, while the overcoming of the latter is foreign to its purpose. For example, if a pulley is employed for the purpose of lifting a weight by means of a rope, a part of the effort employed is spent in overcoming the friction between the pulley and its spindle, and another part is spent in overcoming the rigidity of the rope.

Friction and rigidity in this case are the wasteful resistances, and the weight of the body lifted is the useful resistance.

The distinction between the resistances overcome gives also the distinction between *useful work* and (so-called) *lost work.*

Useful work is that which is performed in overcoming the particular resistance which the machine is designed to overcome, while lost work is that which is spent in overcoming wasteful resistances.

124.] **Efficiency of a Machine.** The ratio of the *useful* work yielded by a machine to the whole amount of work performed by it is called its *efficiency.*

Let W be the work done by the applied forces, W_u the useful and W_l the lost work, when the machine is moving uniformly. Then

$$W = W_u + W_l;$$

and if η denote the efficiency of the machine,

$$\eta = \frac{W_u}{W}.$$

Since some of the work expended in moving the machine must be expended in overcoming wasteful resistances, the efficiency is always less than unity, and the object of most improvements in the machine is to bring its efficiency as near unity as possible.

The *counter-efficiency* is the reciprocal of the efficiency. If the useful work to be performed is given, the amount of work to be expended on the machine is obtained by multiplying the former by the counter-efficiency.

Let P be the effort applied at any point of a machine to perform a given amount, W_u, of useful work; let W_l be the work lost, and let s be the distance through which P drives its point of application in its own direction. Then we have

$$Ps = W_u + W_l.$$

Let P_0 be the force which would perform the same amount of useful work if the wasteful resistances were removed. Then

$$P_0 s = W_u.$$

But $\eta = \frac{W_u}{Ps} = \frac{P_0}{P}$; hence the efficiency is the ratio of the force which would drive the machine against a given useful resistance, if the wasteful resistances were removed, to the force which is actually required to do so. In many cases this definition is useful in practice.

The following consequence regarding efficiency can be at once proved from the principle of Work. In any machine for raising a mass, if the friction in the machine is of itself sufficient to hold the mass suspended, the efficiency is less than $\frac{1}{2}$. If an effort P is required to raise the mass, and an effort P' to sustain it, the efficiency is $\frac{P+P'}{2P}$.

As regards the wasteful resistances in machines, the most noticeable are friction, the rigidity (or rather imperfect flexibility) of ropes, and the vibrations which are produced in the various pieces. Of these the first is that with which alone we shall be concerned. The student who desires information on the experimental laws of the rigidity of ropes may consult Coxe's translation of Weisbach's *Mechanics of Engineering and of the Construction of Machines*, vol. i. p. 363 (New York, 1872).

125.] **Simple Machines.** By simple machines are meant the Lever, the Inclined Plane, the Pulley, the Wheel and Axle, the Screw, and the Wedge. Of these, the Lever, the Inclined Plane, and the Pulley may be considered as distinct in principle, while the others are only combinations of pairs of these three.

126.] **The Lever.** A lever is a solid bar, straight or curved, which is constrained to turn round a fixed axis. This fixed axis is called the *fulcrum* of the lever. It is usual to define three kinds of levers. If the fulcrum is between the effort and the resistance the lever is said to be of the *first kind*; if the resistance acts between the effort and the fulcrum (as in a wheelbarrow, an oar, or a pair of nutcrackers), the lever is of the *second kind*; and if the effort acts between the fulcrum and the resistance (as in the construction of the limbs of animals), the lever is of the *third kind*. In the last kind the effort is always greater than the resistance to be overcome.

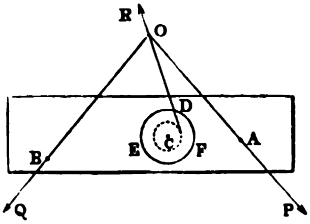

Fig. 151.

To find the efficiency of a lever, the wasteful resistance being friction—

Let the effort P be applied at the point A (Fig. 151) in the direction OA perpendicular to the axis, and the useful resistance at B in the direction OB, also perpendicular to the axis; let EDF be a section of the axis on which the lever turns, made by the plane of P and Q, the contact between the beam and its axis, although it may be very close, being still such that they can be considered as touching along a single line when the machine works. In this case (see Art. 106) the reaction of the axis consists of a single force touching the circle of radius $r \sin \lambda$ concentric with EDF, λ being the angle of friction for the lever and its axis; and since this reaction must also pass through O, its direction is obtained by drawing from this point a tangent to the circle.

Let p and q be the perpendiculars from C, the centre of the axis, on OA and OB, respectively, and let $\omega = \angle AOB$.

Then by moments about C, we have

$$Pp = Qq + Rr \sin \lambda;$$

also

$$R = \sqrt{P^2 + 2PQ \cos \omega + Q^2};$$

$$\therefore \quad Pp = Qq + r \sin \lambda \sqrt{P^2 + 2PQ \cos \omega + Q^2}. \qquad (1)$$

If P_0 is the value of P when friction is removed,

$$P_0 p = Qq, \quad \therefore \quad \eta = \frac{P_0}{P} = \frac{Qq}{Pp}.$$

Substituting $\frac{p}{q}\eta$ for $\frac{Q}{P}$ in (1), we have

$$pq(1-\eta) = r \sin \lambda \sqrt{p^2\eta^2 + 2pq \cos \omega \,.\, \eta + q^2},$$

which gives for the efficiency

$$\eta = \frac{q}{p} \cdot \frac{pq + r^2 \cos \omega \sin^2 \lambda - r \sin \lambda \sqrt{p^2 + 2pq \cos \omega + q^2 - r^2 \sin^2 \omega \sin^2 \lambda}}{q^2 - r^2 \sin^2 \lambda}.$$

If the coefficient of friction is small, we shall have, approximately,

$$\eta = 1 - \frac{\mu r}{pq} \sqrt{p^2 + 2pq \cos \omega + q^2}.$$

If P and Q are parallel, $\omega = 0$, and $\eta = 1 - \mu r\left(\frac{1}{q} + \frac{1}{p}\right)$.

If the lever is of the second kind, and P and Q parallel, $\omega = \pi$, and $\eta = 1 - \mu r\left(\frac{1}{q} - \frac{1}{p}\right)$; and for a lever of the third kind, we find easily in the same circumstances

$$\eta = 1 - \mu r\left(\frac{1}{p} - \frac{1}{q}\right).$$

127.] **The Inclined Plane.** Let an effort, P, whose direction makes an angle θ with a rough inclined plane, be employed to drag a mass of weight Q up the plane. Then if λ is the angle of friction and i the inclination of the plane,

$$P = Q\frac{\sin\,(i+\lambda)}{\cos\,(\theta-\lambda)},$$

$$P_0 = Q\frac{\sin\,i}{\cos\,\theta},$$

$$\therefore\quad \eta = \frac{1+\mu\tan\theta}{1+\mu\cot i}\,.$$

128.] **Fixed and Moveable Pulley.** Let a flexible string pass over a smooth *fixed* pulley (that is, a pulley whose axis is fixed in space), and let a mass of weight W be suspended from one extremity of the string, while a vertical downward force P is applied at the other extremity. Then to raise W we must have $P = W$, and in the uniform working of the machine W is raised exactly as much as the point of application of P is lowered.

Suppose, on the contrary, that one extremity of the string is fixed, that the string passes under a *moveable* pulley from which W is suspended, and that P acts vertically upward at the other extremity of the string. Then evidently $P = \frac{1}{2}W$; hence in the moveable pulley there is a gain in force. But in this case W is raised only *half* as much as the point of application of P ascends. There is, therefore, a loss in the expedition with which the work of raising the weight is performed.

129.] **Systems of Smooth Pulleys.** We shall consider three different arrangements of pulleys, as exemplifying the Principle of Virtual Work.

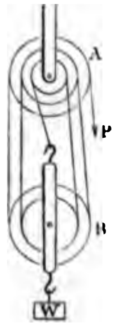

Fig. 152.

I. In the first system there are two blocks, A and B (Fig. 152), the upper of which is fixed and the lower moveable.

Each block contains a number of separate pulleys, of the same diameter usually, each pulley being moveable round the axis of the block in which it is. (The figure represents a section of the blocks made by a plane perpendicular to their axes, and the circumferences of the pulleys are projected on this plane.) A single rope (whose weight is neglected) is attached to the lower block and passes alternately round the pulleys in the upper and under blocks. The portion of rope proceeding from one pulley to the next is called a *ply*. In this arrangement the tension of the rope is throughout constant and equal to P, the force applied at the free extremity. The portion of the rope at which the effort, P, is applied, is called the *tackle-fall.*

Let W be the weight to be lifted, and assume all the plies to be parallel.

Then if n is the number of plies at the lower block, we shall obviously have, neglecting the weight of the block,

$$nP = W.$$

This result follows also by the principle of work. For if p denote the length of the tackle-fall, and x the common length of the plies, we have

$$p + nx = \text{constant},$$

$$\therefore\ dp + ndx = 0.$$

But

$$Pdp + Wdx = 0,$$

$$\therefore\ P = \frac{1}{n}W.$$

II. Suppose each pulley to hang from a fixed beam by a separate rope. Let A (Fig. 153) be the fixed pulley, n the number of moveable pulleys, and x_1, x_2, ... x_n the distances of the centres of these latter from a horizontal plane through the centre of A.

Then, p being the length (AP) of the tackle-fall,

$$2x_1 + p = \text{const.},\quad 2x_2 - x_1 = \text{const.},$$

$$2x_3 - x_2 = \text{const.} \ldots 2x_n - x_{n-1} = \text{const.}$$

Hence $2^n x_n + p = \text{const.}$, therefore

$$2^n dx_n + dp = 0,$$

and

$$Wdx_n + Pdp = 0,$$

$$\therefore\ P = \frac{W}{2^n}.$$

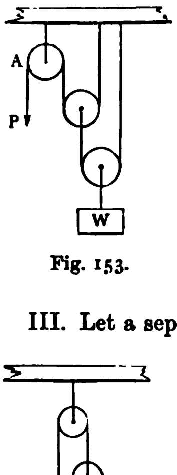

Fig. 153.

III. Let a separate rope pass over each pulley, and let all the ropes be attached to the mass to be raised.

Neglecting the weights of the pulleys and ropes, we shall have, by resolving vertically for the equilibrium of W,

$$W = P(1 + 2 + 2^2 + \ldots + 2^{n-1}),$$

the whole number of pulleys being n; or

$$P = \frac{W}{2^n - 1}.$$

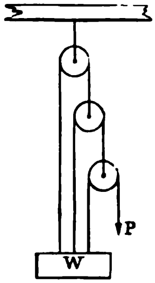

Fig. 154.

The same result follows by the principle of work. For if the distance of W from a horizontal plane through the centre of the fixed pulley is denoted by y, and if the distances of the centres of the pulleys, counting from the fixed one, are $x_1, x_2, \ldots, x_{n-1}$, we have evidently

$$y + x_1 = \text{const.},\quad y + x_2 - 2x_1 = \text{const.} \ldots,\quad y + x_{n-1} - 2x_{n-2} = \text{const.},$$

$$y + p - x_{n-1} = \text{const.}$$

Hence, multiplying the second result by $\frac{1}{2}$, the third by $\frac{1}{2^2}$, &c., and adding, we have $2^{n-1}y+p=$ constant. Now the equation of work is

$$Wdy+Pd(p+x_{n-1})=0,$$

or

$$(W+P)dy+2Pdp=0;$$

and

$$2^{n-1}dy+dp=0,$$

$$\therefore\ P=\frac{W}{2^n-1}.$$

130.] **The Wheel and Axle.** This consists of a horizontal cylinder, *b* (Fig. 155), moveable round two journals (or small cylinders projecting from the centres of its faces), one of which is represented in section at *c*; a wheel, *a*, is rigidly connected with the cylinder, and the journals rotate in fixed bearings. The machine is, in reality, a rigid combination of two pulleys, *a* and *b*, moveable about a common axis, *c*; and its theory is precisely the same as that of the lever. The effort, *P*, is applied at the circumference of the wheel, and the useful resistance, *Q*, at the free extremity of a rope coiled round the axle.

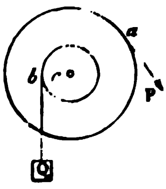

Fig. 155.

All wasteful resistances being neglected, the relation between *P* and *Q* is

$$Pa=Qb,$$

where $a=$ radius of wheel, and $b=$ radius of axle.

The friction of the journal (whose radius is *c*) against its bearing being taken into account, the relation between *P* and *Q* is

$$Pp=Qq+c\sin\lambda\sqrt{P^2+2PQ\cos\omega+Q^2},$$

ω being the angle between the directions of *P* and *Q*, exactly as in Art. 126; and the efficiency is the same as that investigated in the Article on the lever.

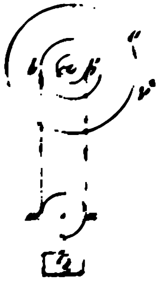

Fig. 156.

Economy of force is attained in the wheel and axle by diminishing *b*, the radius of the axle; but in this way the strength of the machine is diminished. To avoid this disadvantage a *Differential Wheel and Axle* is sometimes employed. In this instrument the axle consists of two cylinders of dif-

b and b' (Fig. 156), and the rope, wound round the former in a sense opposite to that of watch-hand rotation (suppose), leaves it (at the point b in Fig. 155), and, after passing under a moveable pulley to which the mass to be raised is attached, is wound in the opposite sense round the remaining portion (that of radius b') of the axle. The effort P is applied, as before, tangentially to the wheel. For the equilibrium (or uniform motion) of the machine, the tensions of the rope in bm and $b'n$ are each equal to $\frac{1}{2} Q$; and taking moments round the centre of the journal, c, for the equilibrium (or uniform motion) of the rigid system consisting of the wheel and axle alone, we have

$$Pa = \tfrac{1}{2} Q (b - b').$$

Thus, by making the difference $b - b'$ small, the requisite effort can be made as small as we please; but since the amount of *work* to be done is constant, this economy of force is accompanied by a loss in the time of performing the work. For it is easily seen that if the wheel turns through an angle $\delta\theta$, the point of application of P will describe a space $a\delta\theta$, and the suspended mass will be raised through a space $\frac{1}{2}(b - b')\delta\theta$, which latter will be very small if $b - b'$ is very small.

131.] **The Screw.** The screw consists of a right circular cylinder on the convex circumference of which there is a uniform projecting thread, GH (Fig. 158), of a *helical* form.

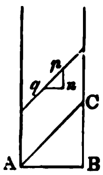

Fig. 157.

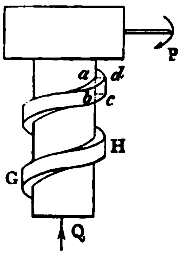

Fig. 158.

The helix is a curve traced on the circumference of a cylinder in the following manner. Take a sheet of paper on which are drawn two indefinite right lines, AB and AC, and let the paper be wound round the cylinder in such a way that the line AB coincides with the circumference of the base; then the other line, AC, will appear on the cylinder in the shape of a spiral curve which is called the *helix*. (Fig. 157 represents a projection of the helix on a plane through the axis of the cylinder.)

A screw with a *rectangular* thread (which is that represented in Fig. 158) is obtained by making a small rectangular area, $abcd$, move so that one side, ab, always coincides with a generating line of the cylinder, the middle point of *ab* describing the helix and the plane of the rectangle always passing through the axis of the cylinder.

If a small *triangle* is used instead of the rectangle, we have a screw with a *triangular* thread.

Let p and q be two points on the indefinite line AC, and draw pn perpendicular to AB and qn parallel to it. Then pq becomes a portion of the arc of the helix, and qn a portion of a section of the cylinder perpendicular to its axis, pn remaining a straight line coinciding with a generator of the cylinder.

Hence the relation holding between the sides of the triangle pqn before the paper was wound round the cylinder will hold also after the winding. But if the angle between AB and AC is i, we have evidently

$$pn = qn \,.\, \tan i,$$
$$pq = qn \,.\, \sec i.$$

The thread GH works in a block on the inner surface of which is cut a groove which is the exact counterpart of the thread. The block in which the groove is cut is often called the *nut*. It is clear, then, that if the screw moves in the nut until the point p of the thread occupies the position q, the axis must move in its own direction through a space pn, and the angular rotation of the screw about its axis is $\frac{qn}{r}$, r being the radius of the cylinder.

Hence, if the angle $\frac{qn}{r}$ through which the screw turns is denoted by ω, we have

$$pn = \omega r \tan i, \quad pq = \omega r \sec i.$$

If $\omega = 2\pi$, or if the screw make a complete revolution, any point on the surface of the screw describes a space $2\pi r \tan i$ parallel to the axis. This is obviously the distance between two portions of the thread measured on a generator, and is called the *pitch* of the screw.

We shall consider the screw as driving a resistance Q applied in the direction of the axis, and the effort, P, as applied in a plane perpendicular to the axis, at the extremity of an arm whose length measured from the centre of the axis is a.

Suppose that the screw rotates through an angle ω. Then the work done by P is $Pa\omega$, and the work done against Q is $Qr\omega \tan i$.

If no work is lost against wasteful resistance, we must have

$$Pa = Qr\tan i.$$

If there is friction between the thread and the groove, let R be the normal pressure on any element of the thread (acting towards the under side of pq in the figure), and μR the friction on this element. Then, in a small angular motion, $\delta\omega$, of the screw the work done against the friction is $\mu R.pq$ (taking pq as an elementary portion of the thread), or $\mu Rr\delta\omega\sec i$. Hence

$$Pa\delta\omega = Qr\delta\omega\tan i + \mu r\delta\omega\sec i\Sigma R,$$

ΣR denoting the sum of the normal reactions at all points of the thread.

But, for the equilibrium of the cylinder, resolving along its axis, we have $Q = \Sigma(R\cos i - \mu R\sin i)$,

or $$Q = (\cos i - \mu\sin i)\Sigma R. \qquad (\alpha)$$

Hence, substituting this value of ΣR in the previous equation,

$$Pa = Qr\tan(i+\lambda),$$

λ being the angle of friction.

This result could have been obtained without the principle of work by combining with (α) the equation of moments round the axis of the screw. By taking moments round the axis, we have

$$Pa = \Sigma(R\sin i + \mu R\cos i)r,$$

or, $$Pa = r(\sin i + \mu\cos i)\Sigma R. \qquad (\beta)$$

Dividing (β) by (α) we obtain the relation between P and Q.

The efficiency of the screw is evidently

$$\frac{\tan i}{\tan(i+\lambda)},$$

which will be a maximum when $i = \frac{\pi}{4} - \frac{\lambda}{2}$.

A good illustration of the action of a screw is furnished by the common coupling-screw for railway carriages. To the axis of the screw whose ends work in nuts, one connected with one carriage and the other with the second carriage, is rigidly attached a rod perpendicularly to the axis, and at the extremity of this rod is attached a heavy ball. If the ball be taken in the hand and swung round, it will cause the screw to rotate and thereby pull the two carriages together in spite of the buffer-springs which tend to keep them apart.

132.] **Prony's Differential Screw.** If h denote the pitch of a screw, the relation between P and Q when friction is neglected is

$$2P\pi a = Qh;$$

therefore economy of force in overcoming a given resistance is gained by making h very small. But it is impossible to do this in practice, and to attain the result desired a differential method is resorted to. Let the screw work in two blocks, A and B (Fig. 159), the first of which is fixed and the second moveable along a fixed groove, n. Let h be the pitch of the thread which works in the block A, and h' the pitch of that which works in the block B. Then one complete revolution of the screw impresses two opposite motions on the block B—one equal to h in the direction in which the screw advances, and the other equal to h' in the opposite direction. If, then, the resistance, Q, is driven by this block, we have by the principle of work

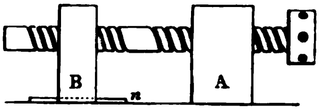

Fig. 159.

$$2P\pi a = Q(h-h'),$$

and the requisite effort will be diminished by diminishing $h-h'$.

133.] **The Wedge.** The wedge is a triangular prism, usually isosceles, which is used (as represented in the figure) for the purpose of separating two bodies, A and B, or parts of the same body which are kept together by some considerable force, molecular or other.

The figure represents a section of the wedge made through the line of action of the effort, P, perpendicular to the axis of the wedge. Suppose that the line of action of P passes through the vertex of the wedge, and that slipping is about to take place; then the total resistances of the surfaces A and B against the wedge will make the angle, λ, of friction with the normals at the points, m and n, where they act; but these points are indeterminate themselves.

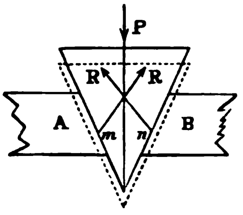

Fig. 160.

To find the efficiency of the wedge. Let the wedge be driven through a vertical space equal to dp, and let $2a$ be its vertical angle. Then the useful work performed is the separation of A and B in directions normal to the faces of the wedge in contact with them; in other words, the useful work is that done by the normal components of the total resistances, R. Now the point m moves vertically down through a space dp, and the projection of this displacement along the normal at m is evidently

$$\sin a \, . \, dp.$$

Hence the work done by the normal components is

$$2R \cos \lambda \sin a dp,$$

and the whole work expended is $P dp$. Hence

$$\eta = \frac{2R \cos\lambda \sin a}{P}.$$

But by resolving vertically for the equilibrium of the wedge, we have

$$P = 2R \sin(a + \lambda);$$

$$\therefore \quad \eta = \frac{\sin a \cos \lambda}{\sin(a+\lambda)} = \frac{\tan a}{\mu + \tan a}.$$

Having given the theory of the simplest machines, we proceed to discuss a few of their most useful forms.

134.] **The Balance.** The balance is a machine which is employed for determining the weights of bodies, and in its common form it may be considered as an instance of a lever of the first kind.

Suppose a rigid body to be suspended from a fixed smooth horizontal axis at a point O (Fig. 161), G being the centre of gravity of the body, at a distance OG ($= h$) from the centre of the axis of suspension, vertically below O when the body rests under the influence of its weight alone.

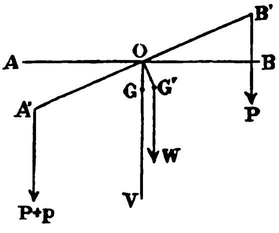

Fig. 161.

Let A and B be two points in the body in the horizontal line through O at equal distances (each $= a$) from O on opposite sides, and suppose two masses whose weights are $P+p$ and P to be suspended from A and B respectively, W being the weight of the rigid body. Then for the position of equilibrium, if θ denotes the angle, AOA', through which the body turns, we have, by moments about O,

$$\tan\theta = \frac{p \, . \, a}{W \, . \, h}. \qquad (1)$$

Such a rigid body may be regarded as a common balance, and it is usually made in the shape of an elongated lozenge-shaped bar or beam, symmetrical on both sides of O, two scale-pans of equal masses being suspended from A and B, the masses P and $P+p$ being placed in these pans.

Sensibility is a desirable property of a balance; that is, for a given difference, p, between the masses suspended from A and B, the deflection θ ought to be as great as it can be consistently with rigidity, etc., in the machine. Now from (1) we see that the greater is the sensibility,

(1) the greater a, i. e., the longer the beam,

(2) the less W, i. e., the lighter the beam,

(3) the less h, i. e., the nearer the centre of gravity to the fulcrum.

Of course h must not be made zero; for if it were, the slightest difference of weight would always make $\theta = \frac{\pi}{2}$.

It appears, therefore, that the sensibility, in the above arrangement, does not depend on the magnitude of the mass in either pan but on the *difference* between these masses.

The sensibility is generally measured by the amount of deflection caused by a difference of weight, p, equal to 1 unit—say that of 1 milligramme in the case of extremely fine and accurate balances such as are used in Physical and Chemical Laboratories.

If the point, O, of support of the beam were not in the same right line as the points, A and B, of suspension of the pans, it would be no longer true that the sensibility depends merely on the difference of the weights of the two compared masses and not on these masses themselves. For, if the arrangement is that represented in Fig. 162, in which AOB is not a right line, and if $OC = c$, we have for the deflection

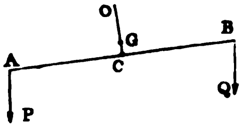

Fig. 162.

$$\tan\theta = \frac{p \, . \, a}{Wh + (P+Q)c},$$

the suspended masses being P and Q, whose difference $= p$.

Friction and rigidity in this case are the wasteful resistances, and the weight of the body lifted is the useful resistance.

The distinction between the resistances overcome gives also the distinction between *useful work* and (so-called) *lost work.*

Useful work is that which is performed in overcoming the particular resistance which the machine is designed to overcome, while lost work is that which is spent in overcoming wasteful resistances.

124.] **Efficiency of a Machine.** The ratio of the *useful* work yielded by a machine to the whole amount of work performed by it is called its *efficiency.*

Let W be the work done by the applied forces, W_u the useful and W_l the lost work, when the machine is moving uniformly. Then

$$W = W_u + W_l;$$

and if η denote the efficiency of the machine,

$$\eta = \frac{W_u}{W}.$$

Since some of the work expended in moving the machine must be expended in overcoming wasteful resistances, the efficiency is always less than unity, and the object of most improvements in the machine is to bring its efficiency as near unity as possible.

The *counter-efficiency* is the reciprocal of the efficiency. If the useful work to be performed is given, the amount of work to be expended on the machine is obtained by multiplying the former by the counter-efficiency.

Let P be the effort applied at any point of a machine to perform a given amount, W_u, of useful work; let W_l be the work lost, and let s be the distance through which P drives its point of application in its own direction. Then we have

$$Ps = W_u + W_l.$$

Let P_0 be the force which would perform the same amount of useful work if the wasteful resistances were removed. Then

$$P_0 s = W_u.$$

But $\eta = \frac{W_u}{Ps} = \frac{P_0}{P}$; hence the efficiency is the ratio of the force which would drive the machine against a given useful resistance, if the wasteful resistances were removed, to the force which is actually required to do so. In many cases this definition is useful in practice.

The following consequence regarding efficiency can be at once proved from the principle of Work. In any machine for raising a mass, if the friction in the machine is of itself sufficient to hold the mass suspended, the efficiency is less than ½. If an effort P is required to raise the mass, and an effort P' to sustain it, the efficiency is $\frac{P+P'}{2P}$.

As regards the wasteful resistances in machines, the most noticeable are friction, the rigidity (or rather imperfect flexibility) of ropes, and the vibrations which are produced in the various pieces. Of these the first is that with which alone we shall be concerned. The student who desires information on the experimental laws of the rigidity of ropes may consult Coxe's translation of Weisbach's *Mechanics of Engineering and of the Construction of Machines*, vol. i. p. 363 (New York, 1872).

125.] **Simple Machines.** By simple machines are meant the Lever, the Inclined Plane, the Pulley, the Wheel and Axle, the Screw, and the Wedge. Of these, the Lever, the Inclined Plane, and the Pulley may be considered as distinct in principle, while the others are only combinations of pairs of these three.

126.] **The Lever.** A lever is a solid bar, straight or curved, which is constrained to turn round a fixed axis. This fixed axis is called the *fulcrum* of the lever. It is usual to define three kinds of levers. If the fulcrum is between the effort and the resistance the lever is said to be of the *first kind*; if the resistance acts between the effort and the fulcrum (as in a wheelbarrow, an oar, or a pair of nutcrackers), the lever is of the *second kind*; and if the effort acts between the fulcrum and the resistance (as in the construction of the limbs of animals), the lever is of the *third kind*. In the last kind the effort is always greater than the resistance to be overcome.

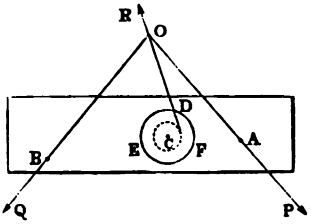

Fig. 151.

To find the efficiency of a lever, the wasteful resistance being friction—

Let the effort P be applied at the point A (Fig. 151) in the direction OA perpendicular to the axis, and the useful resistance at B in the direction OB, also perpendicular to the axis; let EDF be a section of the axis on which the lever turns, made by the plane of P and Q, the contact between the beam and its axis, although it may be very close, being still such that they can be considered as touching along a single line when the machine works. In this case (see Art. 106) the reaction of the axis consists of a single force touching the circle of radius $r \sin \lambda$ concentric with EDF, λ being the angle of friction for the lever and its axis; and since this reaction must also pass through O, its direction is obtained by drawing from this point a tangent to the circle.

Let p and q be the perpendiculars from C, the centre of the axis, on OA and OB, respectively, and let $\omega = \angle AOB$.

Then by moments about C, we have

$$Pp = Qq + Rr \sin \lambda;$$

also

$$R = \sqrt{P^2 + 2PQ \cos \omega + Q^2};$$

$$\therefore \quad Pp = Qq + r \sin \lambda \sqrt{P^2 + 2PQ \cos \omega + Q^2}. \tag{1}$$

If P_0 is the value of P when friction is removed,

$$P_0 p = Qq, \quad \therefore \quad \eta = \frac{P_0}{P} = \frac{Qq}{Pp}.$$

Substituting $\frac{p}{q}\eta$ for $\frac{Q}{P}$ in (1), we have

$$pq(1-\eta) = r \sin \lambda \sqrt{p^2\eta^2 + 2pq \cos \omega \,.\, \eta + q^2},$$

which gives for the efficiency

$$\eta = \frac{q}{p} \cdot \frac{pq + r^2 \cos \omega \sin^2 \lambda - r \sin \lambda \sqrt{p^2 + 2pq \cos \omega + q^2 - r^2 \sin^2 \omega \sin^2 \lambda}}{q^2 - r^2 \sin^2 \lambda}.$$

If the coefficient of friction is small, we shall have, approximately,

$$\eta = 1 - \frac{\mu r}{pq} \sqrt{p^2 + 2pq \cos \omega + q^2}.$$

If P and Q are parallel, $\omega = 0$, and $\eta = 1 - \mu r\left(\frac{1}{q} + \frac{1}{p}\right)$.

If the lever is of the second kind, and P and Q parallel, $\omega = \pi$, and $\eta = 1 - \mu r\left(\frac{1}{q} - \frac{1}{p}\right)$; and for a lever of the third kind, we find easily in the same circumstances

$$\eta = 1 - \mu r\left(\frac{1}{p} - \frac{1}{q}\right).$$

127.] **The Inclined Plane.** Let an effort, P, whose direction makes an angle θ with a rough inclined plane, be employed to drag a mass of weight Q up the plane. Then if λ is the angle of friction and i the inclination of the plane,

$$P = Q\frac{\sin(i+\lambda)}{\cos(\theta-\lambda)},$$

$$P_0 = Q\frac{\sin i}{\cos\theta},$$

$$\therefore \quad \eta = \frac{1+\mu\tan\theta}{1+\mu\cot i}.$$

128.] **Fixed and Moveable Pulley.** Let a flexible string pass over a smooth *fixed* pulley (that is, a pulley whose axis is fixed in space), and let a mass of weight W be suspended from one extremity of the string, while a vertical downward force P is applied at the other extremity. Then to raise W we must have $P = W$, and in the uniform working of the machine W is raised exactly as much as the point of application of P is lowered.

Suppose, on the contrary, that one extremity of the string is fixed, that the string passes under a *moveable* pulley from which W is suspended, and that P acts vertically upward at the other extremity of the string. Then evidently $P = \frac{1}{2} W$; hence in the moveable pulley there is a gain in force. But in this case W is raised only *half* as much as the point of application of P ascends. There is, therefore, a loss in the expedition with which the work of raising the weight is performed.

129.] **Systems of Smooth Pulleys.** We shall consider three different arrangements of pulleys, as exemplifying the Principle of Virtual Work.

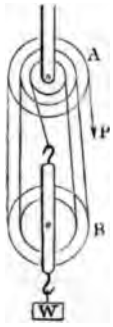

Fig. 152.

I. In the first system there are two blocks, A and B (Fig. 152), the upper of which is fixed and the lower moveable.

Each block contains a number of separate pulleys, of the same diameter usually, each pulley being moveable round the axis of the block in which it is. (The figure represents a section of the blocks made by a plane perpendicular to their axes, and the circumferences of the pulleys are projected on this plane.) A single rope (whose weight is neglected) is attached to the lower block and passes alternately round the pulleys in the upper and under blocks. The portion of rope proceeding from one pulley to the next is called a *ply*. In this arrangement the tension of the rope is throughout constant and equal to P, the force applied at the free extremity. The portion of the rope at which the effort, P, is applied, is called the *tackle-fall.*

Let W be the weight to be lifted, and assume all the plies to be parallel.

Then if n is the number of plies at the lower block, we shall obviously have, neglecting the weight of the block,

$$nP = W.$$

This result follows also by the principle of work. For if p denote the length of the tackle-fall, and x the common length of the plies, we have

$$p + nx = \text{constant},$$

$$\therefore \quad dp + ndx = 0.$$

But

$$Pdp + Wdx = 0,$$

$$\therefore \quad P = \frac{1}{n} W.$$

II. Suppose each pulley to hang from a fixed beam by a separate rope. Let A (Fig. 153) be the fixed pulley, n the number of moveable pulleys, and $x_1, x_2, \ldots x_n$ the distances of the centres of these latter from a horizontal plane through the centre of A.

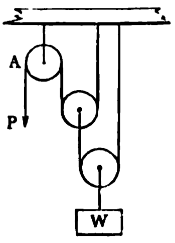

Fig. 153.

Then, p being the length (AP) of the tackle-fall,

$$2x_1 + p = \text{const.}, \quad 2x_2 - x_1 = \text{const.},$$

$$2x_3 - x_2 = \text{const.} \ldots 2x_n - x_{n-1} = \text{const.}$$

Hence $2^n x_n + p = \text{const.}$, therefore

$$2^n dx_n + dp = 0,$$

and

$$Wdx_n + Pdp = 0,$$

$$\therefore \quad P = \frac{W}{2^n}.$$

III. Let a separate rope pass over each pulley, and let all the ropes be attached to the mass to be raised.

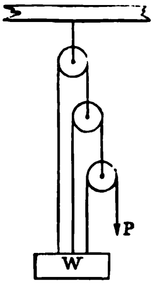

Fig. 154.

Neglecting the weights of the pulleys and ropes, we shall have, by resolving vertically for the equilibrium of W,

$$W = P(1 + 2 + 2^2 + \ldots + 2^{n-1}),$$

the whole number of pulleys being n; or

$$P = \frac{W}{2^n - 1}.$$

The same result follows by the principle of work. For if the distance of W from a horizontal plane through the centre of the fixed pulley is denoted by y, and if the distances of the centres of the pulleys, counting from the fixed one, are $x_1, x_2, \ldots, x_{n-1}$, we have evidently

$$y + x_1 = \text{const.}, \quad y + x_2 - 2x_1 = \text{const.} \ldots, \quad y + x_{n-1} - 2x_{n-2} = \text{const.},$$

$$y + p - x_{n-1} = \text{const.}$$

Hence, multiplying the second result by $\frac{1}{2}$, the third by $\frac{1}{2^2}$, &c., and adding, we have $2^{n-1}y+p=$ constant. Now the equation of work is

$$Wdy+Pd(p+x_{n-1})=0,$$

or

$$(W+P)dy+2Pdp=0;$$

and

$$2^{n-1}dy+dp=0,$$

$$\therefore \quad P=\frac{W}{2^n-1}.$$

130.] **The Wheel and Axle.** This consists of a horizontal cylinder, *b* (Fig. 155), moveable round two journals (or small cylinders projecting from the centres of its faces), one of which is represented in section at *c*; a wheel, *a*, is rigidly connected with the cylinder, and the journals rotate in fixed bearings. The machine is, in reality, a rigid combination of two pulleys, *a* and *b*, moveable about a common axis, *c*; and its theory is precisely the same as that of the lever. The effort, *P*, is applied at the circumference of the wheel, and the useful resistance, *Q*, at the free extremity of a rope coiled round the axle.

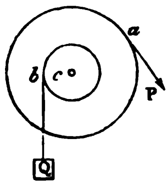

Fig. 155.

All wasteful resistances being neglected, the relation between *P* and *Q* is

$$Pa=Qb,$$

where $a=$ radius of wheel, and $b=$ radius of axle.

The friction of the journal (whose radius is *c*) against its bearing being taken into account, the relation between *P* and *Q* is

$$Pp=Qq+c\sin\lambda\sqrt{P^2+2PQ\cos\omega+Q^2},$$

ω being the angle between the directions of *P* and *Q*, exactly as in Art. 126; and the efficiency is the same as that investigated in the Article on the lever.

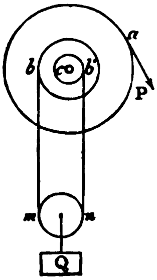

Fig. 156.

Economy of force is attained in the wheel and axle by diminishing *b*, the radius of the axle; but in this way the strength of the machine is diminished. To avoid this disadvantage a *Differential Wheel and Axle* is sometimes employed. In this instrument the axle consists of two cylinders of radii

b and b' (Fig. 156), and the rope, wound round the former in a sense opposite to that of watch-hand rotation (suppose), leaves it (at the point b in Fig. 155), and, after passing under a moveable pulley to which the mass to be raised is attached, is wound in the opposite sense round the remaining portion (that of radius b') of the axle. The effort P is applied, as before, tangentially to the wheel. For the equilibrium (or uniform motion) of the machine, the tensions of the rope in bm and $b'n$ are each equal to $\frac{1}{2} Q$; and taking moments round the centre of the journal, c, for the equilibrium (or uniform motion) of the rigid system consisting of the wheel and axle alone, we have

$$Pa = \tfrac{1}{2} Q (b-b').$$

Thus, by making the difference $b-b'$ small, the requisite effort can be made as small as we please; but since the amount of *work* to be done is constant, this economy of force is accompanied by a loss in the time of performing the work. For it is easily seen that if the wheel turns through an angle $\delta\theta$, the point of application of P will describe a space $a\delta\theta$, and the suspended mass will be raised through a space $\frac{1}{2}(b-b')\delta\theta$, which latter will be very small if $b-b'$ is very small.

131.] **The Screw.** The screw consists of a right circular cylinder on the convex circumference of which there is a uniform projecting thread, GH (Fig. 158), of a *helical* form.

The helix is a curve traced on the circumference of a cylinder in the following manner. Take a sheet of paper on which are drawn two indefinite right lines, AB and AC, and let the paper be wound round the cylinder in such a way that the line AB coincides with the circumference of the base; then the other line, AC, will appear on the cylinder in the shape of a spiral curve which is called the *helix*. (Fig. 157 represents a projection of the helix on a plane through the axis of the cylinder.)

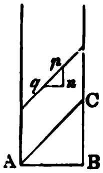

Fig. 157.

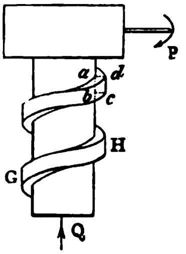

Fig. 158.

A screw with a *rectangular* thread (which is that represented in Fig. 158) is obtained by making a small rectangular area, *abcd*, move so that one side, *ab*, always coincides with a generating line of the cylinder, the middle point of ab describing the helix and the plane of the rectangle always passing through the axis of the cylinder.

If a small *triangle* is used instead of the rectangle, we have a screw with a *triangular* thread.

Let p and q be two points on the indefinite line AC, and draw pn perpendicular to AB and qn parallel to it. Then pq becomes a portion of the arc of the helix, and qn a portion of a section of the cylinder perpendicular to its axis, pn remaining a straight line coinciding with a generator of the cylinder.

Hence the relation holding between the sides of the triangle pqn before the paper was wound round the cylinder will hold also after the winding. But if the angle between AB and AC is i, we have evidently

$$pn = qn \,.\, \tan i,$$
$$pq = qn \,.\, \sec i.$$

The thread GH works in a block on the inner surface of which is cut a groove which is the exact counterpart of the thread. The block in which the groove is cut is often called the *nut*. It is clear, then, that if the screw moves in the nut until the point p of the thread occupies the position q, the axis must move in its own direction through a space pn, and the angular rotation of the screw about its axis is $\frac{qn}{r}$, r being the radius of the cylinder.

Hence, if the angle $\frac{qn}{r}$ through which the screw turns is denoted by ω, we have

$$pn = \omega r \tan i, \quad pq = \omega r \sec i.$$

If $\omega = 2\pi$, or if the screw make a complete revolution, any point on the surface of the screw describes a space $2\pi r \tan i$ parallel to the axis. This is obviously the distance between two portions of the thread measured on a generator, and is called the *pitch* of the screw.

We shall consider the screw as driving a resistance Q applied in the direction of the axis, and the effort, P, as applied in a plane perpendicular to the axis, at the extremity of an arm whose length measured from the centre of the axis is a.

Suppose that the screw rotates through an angle ω. Then the work done by P is $Pa\omega$, and the work done against Q is $Qr\omega \tan i$.

If no work is lost against wasteful resistance, we must have

$$Pa = Qr\tan i.$$

If there is friction between the thread and the groove, let R be the normal pressure on any element of the thread (acting towards the under side of pq in the figure), and μR the friction on this element. Then, in a small angular motion, $\delta\omega$, of the screw the work done against the friction is $\mu R . pq$ (taking pq as an elementary portion of the thread), or $\mu R r \delta\omega \sec i$. Hence

$$Pa\delta\omega = Qr\delta\omega\tan i + \mu r\delta\omega\sec i \Sigma R,$$

ΣR denoting the sum of the normal reactions at all points of the thread.

But, for the equilibrium of the cylinder, resolving along its axis, we have

$$Q = \Sigma(R\cos i - \mu R\sin i),$$

or

$$Q = (\cos i - \mu\sin i)\Sigma R. \qquad (a)$$

Hence, substituting this value of ΣR in the previous equation,

$$Pa = Qr\tan(i+\lambda),$$

λ being the angle of friction.

This result could have been obtained without the principle of work by combining with (a) the equation of moments round the axis of the screw. By taking moments round the axis, we have

$$Pa = \Sigma(R\sin i + \mu R\cos i)r,$$

or,

$$Pa = r(\sin i + \mu\cos i)\Sigma R. \qquad (\beta)$$

Dividing (β) by (a) we obtain the relation between P and Q.

The efficiency of the screw is evidently

$$\frac{\tan i}{\tan(i+\lambda)},$$

which will be a maximum when $i = \frac{\pi}{4} - \frac{\lambda}{2}$.

A good illustration of the action of a screw is furnished by the common coupling-screw for railway carriages. To the axis of the screw whose ends work in nuts, one connected with one carriage and the other with the second carriage, is rigidly attached a rod perpendicularly to the axis, and at the extremity of this rod is attached a heavy ball. If the ball be taken in the hand and swung round, it will cause the screw to rotate and thereby pull the two carriages together in spite of the buffer-springs which tend to keep them apart.

132.] **Prony's Differential Screw.** If h denote the pitch of a screw, the relation between P and Q when friction is neglected is

$$2P\pi a = Qh;$$

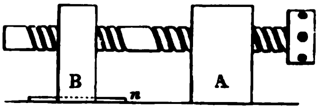

Fig. 159.

therefore economy of force in overcoming a given resistance is gained by making h very small. But it is impossible to do this in practice, and to attain the result desired a differential method is resorted to. Let the screw work in two blocks, A and B (Fig. 159), the first of which is fixed and the second moveable along a fixed groove, n. Let h be the pitch of the thread which works in the block A, and h' the pitch of that which works in the block B. Then one complete revolution of the screw impresses two opposite motions on the block B—one equal to h in the direction in which the screw advances, and the other equal to h' in the opposite direction. If, then, the resistance, Q, is driven by this block, we have by the principle of work

$$2P\pi a = Q(h-h'),$$

and the requisite effort will be diminished by diminishing $h-h'$.

133.] **The Wedge.** The wedge is a triangular prism, usually isosceles, which is used (as represented in the figure) for the purpose of separating two bodies, A and B, or parts of the same body which are kept together by some considerable force, molecular or other.

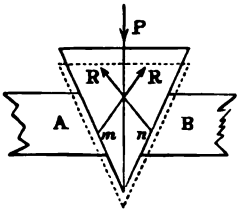

Fig. 160.

The figure represents a section of the wedge made through the line of action of the effort, P, perpendicular to the axis of the wedge. Suppose that the line of action of P passes through the vertex of the wedge, and that slipping is about to take place; then the total resistances of the surfaces A and B against the wedge will make the angle, λ, of friction with the normals at the points, m and n, where they act; but these points are indeterminate themselves.

To find the efficiency of the wedge. Let the wedge be driven through a vertical space equal to dp, and let $2a$ be its vertical angle. Then the useful work performed is the separation of A and B in directions normal to the faces of the wedge in contact with them; in other words, the useful work is that done by the normal components of the total resistances, R. Now the point m moves vertically down through a space dp, and the projection of this displacement along the normal at m is evidently

$$\sin a \,.\, dp.$$

Hence the work done by the normal components is

$$2R\cos\lambda\,\sin a\,dp,$$

and the whole work expended is Pdp. Hence

$$\eta = \frac{2R\cos\lambda\,\sin a}{P}\,.$$

But by resolving vertically for the equilibrium of the wedge, we have

$$P = 2R\sin(a+\lambda);$$

$$\therefore\quad \eta = \frac{\sin a\,\cos\lambda}{\sin(a+\lambda)} = \frac{\tan a}{\mu+\tan a}\,.$$

Having given the theory of the simplest machines, we proceed to discuss a few of their most useful forms.

134.] **The Balance.** The balance is a machine which is employed for determining the weights of bodies, and in its common form it may be considered as an instance of a lever of the first kind.

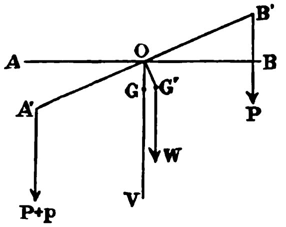

Fig. 161.

Suppose a rigid body to be suspended from a fixed smooth horizontal axis at a point O (Fig. 161), G being the centre of gravity of the body, at a distance $OG\,(=h)$ from the centre of the axis of suspension, vertically below O when the body rests under the influence of its weight alone.

Let A and B be two points in the body in the horizontal line through O at equal distances (each $= a$) from O on opposite sides, and suppose two masses whose weights are $P+p$ and P to be suspended from A and B respectively, W being the weight of the rigid body. Then for the position of equilibrium, if θ denotes the angle, AOA', through which the body turns, we have, by moments about O,

$$\tan\theta = \frac{p \,.\, a}{W \,.\, h}. \tag{1}$$

Such a rigid body may be regarded as a common balance, and it is usually made in the shape of an elongated lozenge-shaped bar or beam, symmetrical on both sides of O, two scale-pans of equal masses being suspended from A and B, the masses P and $P+p$ being placed in these pans.

Sensibility is a desirable property of a balance; that is, for a given difference, p, between the masses suspended from A and B, the deflection θ ought to be as great as it can be consistently with rigidity, etc., in the machine. Now from (1) we see that the greater is the sensibility,

(1) the greater a, i. e., the longer the beam,

(2) the less W, i. e., the lighter the beam,

(3) the less h, i. e., the nearer the centre of gravity to the fulcrum.

Of course h must not be made zero; for if it were, the slightest difference of weight would always make $\theta = \frac{\pi}{2}$.

It appears, therefore, that the sensibility, in the above arrangement, does not depend on the magnitude of the mass in either pan but on the *difference* between these masses.

The sensibility is generally measured by the amount of deflection caused by a difference of weight, p, equal to 1 unit—say that of 1 milligramme in the case of extremely fine and accurate balances such as are used in Physical and Chemical Laboratories.

If the point, O, of support of the beam were not in the same right line as the points, A and B, of suspension of the pans, it would be no longer true that the sensibility depends merely on the difference of the weights of the two compared masses and not on these masses themselves. For, if the arrangement is that represented in Fig. 162, in which AOB is not a right line, and if $OC = c$, we have for the deflection

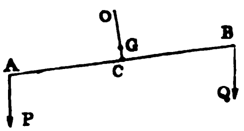

Fig. 162.

$$\tan\theta = \frac{p \,.\, a}{Wh + (P+Q)c},$$

the suspended masses being P and Q, whose difference $= p$.

Now this equation shows that for a given difference, p, the deflection is less than it would be for the balance if c were made zero; i.e., if O were at C.

Lightness in the beam is attained by making it in the shape of a very elongated lozenge, the central portion of which is nearly all cut out, its sides being, however, braced together by light brass cross-pieces, as represented in Fig. 163, which shows only a few portions of the delicate form of the instrument, as it is used for the very minute weighings in laboratories.

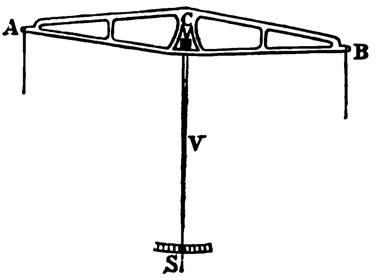

Fig. 163.

The axis of suspension is, in such an instrument, a knife edge (represented vertically under C) which rests on a fixed horizontal plane of agate. The deflections of the beam from the vertical are indicated by a very long pointer, V, connected with the beam, the end of this pointer moving from right to left just in front of a fixed vertical scale, S, divided into a large number of close divisions.

Absolute equilibrium between the masses in the pans is scarcely ever attained; but even if these masses were equal, some slight disturbance might set the pointer swinging to right and left of its zero position, and if the amplitudes of these swings were equal, we could conclude that the masses in the pans are equal. Hence it is obviously desirable that the *time* of these small oscillations should be as small as possible. The determination of the time of a small oscillation depends on kinetical principles, and we must assume the result here.

If k is the radius of gyration of the beam about the axis of suspension, g the acceleration of a body falling freely in vacuo near the Earth's surface, and if two equal, or very nearly equal, masses, P, are suspended in the pans, the time, t, of a small oscillation is given by the equation

$$t = \pi \sqrt{\frac{Wk^2 + 2Pa^2}{Whg}}; \qquad (2)$$

so that if t is to be very small, i. e., the oscillations rapid, a should be small and h large. These conditions are therefore the reverse of those for sensibility, and nothing but a compromise can be effected between them.

Two methods of *exact weighing*—i. e., weighing to within a milligramme, or even one-tenth of it—are in use, namely the *counterpoise method*, or method of Borda, and the *method of Gauss.*

In the first method a mass of shot or sand (the amount of which we need not know) is put into one pan, while the body to be weighed is put into the other, and with this body are put in known weights (multiples of a gramme, decigramme, etc.) until equilibrium of the pointer is attained either in the middle of the scale S or in some position near it. The body to be weighed is now removed from its pan, and its place is occupied by known masses (grammes, etc.) until the same position of equilibrium (or one near it) of the pointer is again attained. Obviously the mass of the body to be weighed must then be equal to the sum of the masses which have taken its place and which produce the same effect as the body did.

The method of Gauss consists in weighing the body first in one pan and then in the other, and taking the weight of the body equal to the geometric mean of the two (very nearly equal) amounts which seem to represent it. If the arms, OA, OB, of the balance were absolutely equal in length, the body would appear to weigh the same in both pans; but very slight differences between the arms may arise from various causes, and we shall therefore assume the arms to be of different lengths, $OA = a$, $OB = b$. Let M be the true weight of the body. When M is suspended from A, let P_1 be the weight of the known mass which must be suspended from B to bring the pointer to rest in the vertical position; that is, P_1 is the apparent weight of the body, and it appears as if $M = P_1$. The true relation, however, between M and P_1 is

$$M \cdot a = P_1 \cdot b. \tag{3}$$

Again, when M is suspended from B, let P_2 be its apparent weight.

Then

$$M \cdot b = P_2 \cdot a. \tag{4}$$

From these equations we have, by multiplication,

$$M = \sqrt{P_1 \cdot P_2}, \tag{5}$$

which proves the result given above. In practice, since the difference between the apparent weights is extremely minute, it is usual to take

$$M = \tfrac{1}{2}(P_1 + P_2), \tag{6}$$

i. e., the arithmetic, instead of the geometric, mean, the two means being very nearly the same.

For a full description of the practical use and details of the balance the student may very profitably consult the *Lessons in Elementary Practical Physics* by Stewart and Gee, (Vol. I).

135.] **Roberval's Balance.** Roberval's Balance is an excellent illustration of the principle of work.

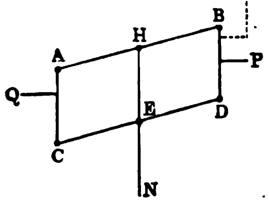

Fig. 164.

Two equal bars, AB and CD (Fig. 164), revolve round axes through their middle points, H and E, which are fixed in a vertical support, HN; these bars are connected by smooth joints to two equal bars, AC and BD, and to these latter bars are rigidly attached two plates or scale pans, P and Q, the points of attachment being any whatever, and one or both of the plates may lie *towards* the vertical support, or away from it (as Fig. 164).

Suppose P and Q to be the weights of two masses placed in the pans P and Q, respectively. Then if for any displacement of the bars round the points H and E, the pans describe vertical spaces p and q, respectively, we shall have for equilibrium

$$Pp - Qq = 0.$$

Now, the bars AC and BD, being always parallel to the fixed line HE, will be always vertical, and the vertical space through which one moves up is obviously equal to that through which the other moves down. Hence $p = q$, and we have for equilibrium

$$P = Q,$$

whatever be the lengths of the pans (provided their weights are neglected), whatever be their points of attachment to BD and AC, and whatever the points in the pans at which P and Q are placed.

If the weights of the pans are taken into account, the same results follow if they are of equal weight.

If the pan P were replaced by the pan P', and the weight P placed at P', the other pan, Q, remaining unchanged, and the weights of the pans being either equal or neglected, equilibrium would still subsist—a result which seems at first sight very strange.

If the lengths AH and HB, CE and ED are not equal, it is easy to prove that $\frac{p}{q} = \frac{HB}{HA}$, and the condition of equilibrium is

$$P.HB = Q.HA.$$

136.] **Balance of Quintenz.** This is a compound balance formed of a combination of several levers, and is used for weighing very heavy loads. This machine also furnishes an admirable example of the principle of work.

AB (Fig. 165) is a lever moveable about its fixed extremity, *A*; *MN* is another lever moveable about a fulcrum, *F*, fixed at its middle point; *CD* is a moveable platform, which receives the load *Q*, whose weight is to be found; this platform is connected with the lever *MN* by a rigid vertical bar, *DI*, articulated at *D* and *I*; and the platform further rests against the lever, *AB*, by an edge of contact at a fixed point, *H*, on the latter; finally, the two levers are connected by a rigid vertical bar, *BM*, articulated to both.

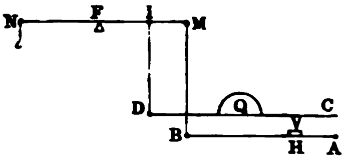

Fig. 165.

The weight, *P*, employed to measure *Q* is applied to the upper lever at *N*. Let the system receive any slight displacement, then the lever, *AB*, will turn round *A* through an angle $\delta\theta$, suppose, and the lever *MN* will turn round *F* through an angle $\delta\phi$.

We shall arrange the dimensions of the machine in such a manner that the platform, *CD*, may remain horizontal in the displacement. The vertical descent of the point *H* is evidently $AH.\delta\theta$, and this is also the vertical descent of the point in the platform above *H*.

The vertical descent of the point *D* is the same as that of *I*, and this latter is obviously $FI.\delta\phi$; hence if the platform remains horizontal, $FI.\delta\phi = AH.\delta\theta.$

Again, the vertical descent of *M* is the same as that of *B*: or

$$FM.\delta\phi = AB.\delta\theta.$$

Hence from these equations we have

$$\frac{MF}{FI} = \frac{BA}{AH},$$

which is the condition for the horizontality of the platform. Denote $\frac{BA}{AH}$ by n. The equation of work is obviously

$$P \times \text{descent of } N = Q \times \text{descent of } D,$$

$$\therefore \quad P.NF\delta\phi = Q.FI\delta\phi,$$

$$\therefore \quad P = \frac{1}{n}Q,$$

or the result is the same as if Q were suspended from the point I of the upper lever.

Loads placed on the platform may all be weighed by means of a constant weight, P, by merely moving the point of suspension of this latter along the arm NF; thus, if P is suspended from the point K between N and F, we shall have

$$\frac{P}{Q} = \frac{FI}{FK}.$$

137.] **Toothed Wheels.** Motion may be transferred from one point to another and work done by means of a combination of toothed wheels, each one of which drives the next one in the series. The discussion of this kind of machinery possesses great geometrical elegance; but the space at our disposal renders it impossible to do more than give a slight sketch of the simplest case—that in which the axes of the wheels are all parallel.

For the investigation of the proper forms of teeth, the student is referred to Willis's *Principles of Mechanism*, Collignon's *Statique*, and Resal's *Mécanique Générale*.

Fig. 166 represents a toothed wheel, A_1, moveable round a horizontal axis, ab; the effort, P, is applied by means of a handle, cd, which, when turned, causes the axis ab to rotate in its bearings at a and b and to turn the wheel A_1; this wheel causes another, B_1, in contact with it, to rotate round a horizontal axis which also moves in fixed bearings at its extremities; on this latter axis is fixed another wheel A_2, whose rotation in like manner turns B_2 on its axis, which in the figure is the axis of a cylinder to which the resistance, Q, is attached.

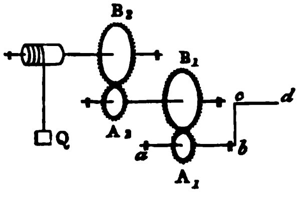

Fig. 166.

Suppose that there are n wheels, $A_1, A_2, \dots A_n$, whose radii are $a_1, a_2, \dots a_n$, and n wheels $B_1, B_2, \dots B_n$, whose radii are $b_1, b_2, \dots b_n$; and let $br = p$, and the radius of the cylinder (or wheel) to which Q is attached $= q$. Then, if ω_1 is the angle through which the radius br revolves, the effort being always applied tangentially to the circle described by its point of application, the work expended is

$$Pp\omega_1:$$

and if ω_n is the angle through which, in the same time, the cylinder rotates, the weight Q will be raised through a distance $q\omega_n$, and the work done against the resistance is

$$Qq\omega_n.$$

Supposing then that no work is lost either by the friction of the axes in their bearings, or by the friction of the teeth against each other, we must have

$$Pp\omega_1 = Qq\omega_n,$$

when the machine is moving uniformly.

To determine the kinematical relation between ω_1 and ω_n, let the angle through which B_1 turns be ω_2. Then, since the distances described by the points of A_1 and B_1 which are in contact are the same, $a_1\omega_1 = b_1\omega_2$. Also if ω_3 is the angle through which B_2 turns, we have $a_2\omega_2 = b_2\omega_3$. Proceeding in this way, we have, by multiplying the corresponding sides of these equations together, $$a_1 a_2 \dots a_n \cdot \omega_1 = b_1 b_2 \dots b_n \cdot \omega_n.$$

Hence from (1) and (2),

$$\frac{Q}{P} = \frac{p}{q} \cdot \frac{b_1 b_2 \dots b_n}{a_1 a_2 \dots a_n}.$$

For the calculation of the work lost by the friction of the teeth among themselves see Collignon's *Statique*, p. 468.

CHAPTER IX.

DETERMINATION OF MUTUAL REACTIONS OF PARTS OF A SYSTEM.

138.] **Action and Reaction.** If in any system of bodies, connected in any manner, A and B are two bodies in contact between which an action of some kind is exercised; then, whatever be the forces with which the body A acts upon the body B, the very same forces, reversed in directions, will constitute the action of B on A. Let the whole system of forces acting on A, excluding those produced by B, be denoted by (P), and let the forces constituting the action of B on A be denoted by (R); then we may sever the connexion between A and B, provided that we have other means of producing on A the system of forces (R). In the same way, if (Q) denote the whole system of forces acting on B, those constituting the action of A on it being excluded, the body, B, may be severed from A provided that we have the means of producing a system of forces $(-R)$ on B, $(-R)$ denoting a system of forces obtained by reversing the direction and preserving the magnitude of every force in (R).

For example, the beam CD (Fig. 126) may be severed from the other beam along any section, CB, provided that there be introduced on CD either the single force R acting through A, or the complex system of tensile and compressive forces which act at the section CB. This equality of magnitude and oppositeness of direction of the forces existing between two distinct bodies in contact, or between ideally severed portions of the same body, is sometimes spoken of as the principle of *the equality of Action and Reaction*; but this is by no means the whole of the Newtonian principle called by this name; for Newton specifies several senses in which the terms Action and Reaction can be taken, and in discussing one of them he has explicitly anticipated, in great part, the principle of the Conservation of Energy—as has been pointed out by Thomson and Tait.

139.] **Examples of Internal Action.** The cases which we shall consider in this chapter are those in which the action between two portions of a system ideally severed consists of a single force. The simplest example of such action occurs when a single point of one body rests against the surface of another, the bodies being either rough or smooth. If the bodies are smooth, the action between them consists of a single force which is *normal to the surface of contact*; and if rough, the action is still a single force which is not necessarily normal to this surface. In all cases in which smooth spherical joints or hinges are concerned, the action exercised on bodies connected by them consists of a single force passing through the centre of the joint. When rough joints are used, the action will generally consist of a single force acting somewhere outside the joint; or of a couple and a force acting at the joint; or, possibly, of a couple alone. The tension of a string is also an instance of internal action, and its nature has been already explained in Chap. II.

Again, if we ideally separate into two portions by an arbitrary surface a mass of a perfect fluid in equilibrium, the action of one portion on the other over a small area of the ideally separating surface will consist of a single force acting normally on the area. And we may always treat as a separate body any portion whatever of a fluid in equilibrium*, *provided that we produce along the surface of this ideally separated portion all the forces which are actually produced on it by the fluid with which it was surrounded.* This is what we have already enunciated as *the principle of separate equilibrium* (p. 32). It is by such separate consideration of portions of a fluid that we arrive at a knowledge of its internal forces or pressures. For example, if a heavy fluid, whether compressible or incompressible, of uniform or varying density, be contained in a vessel, we can prove that the pressure is the same at all points, P, Q, in the same horizontal plane. For, isolate in imagination a horizontal cylindrical column of the fluid, having small vertical and equal areas at P and Q for extremities, from the rest of the fluid. Then, we may treat the cylinder of fluid PQ as a separate body, provided that, in addition to the external force (gravity) acting on it, we introduce the forces which it actually experienced from the surrounding fluid. Now these forces consist of normal pressures, p and q, on the areas at P and Q, together with normal pressures all over its curved surface, these latter being all at right angles to the axis PQ. If now we resolve horizontally all the forces acting on the cylinder, we get

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* It is usually said that we may, under the above condition, imagine any portion of the fluid *to become solidified*; but the imagined *solidification* is not only wholly unnecessary but misleading.

$$p - q = 0, \text{ or } p = q.$$

This demonstration shows, moreover, that in the case of a heavy viscous or imperfect fluid, the pressures are not necessarily equal at all points in the same horizontal plane.

For, in this case, the action of the rest of the fluid on PQ does not necessarily consist of forces normal to its surface, but of oblique forces. Hence the horizontal component of the pressure at P is not equal to the horizontal component at Q; the difference between them is equal to the sum of the horizontal components of the oblique forces.

The importance of keeping such considerations in view may be illustrated by the following example from Hydrostatics.

A conical vessel is filled with water through an aperture at the vertex. From Hydrostatical principles it follows that the pressure on the base of the cone is equal to the weight of a *cylindrical* column of water, standing on the base, and having a height equal to that of the cone; that is, the pressure on the base is much greater than the weight of the water contained in the cone. Now if we imagine the water to become solidified, the curved surface of the cone may be removed, and the pressure on the base will be equal to the weight of the ice, that is, the weight of the water in the cone. An apparent discrepancy is the result. But if we attend to the proviso that in the separate consideration of the equilibrium of any portion of a system, solid or fluid, we must produce upon the isolated portion all the forces which were originally produced upon it by the neighbouring portions of the solid or fluid, the difficulty disappears. In the fluid state the liquid in contact with the curved surface of the cone was pressed normally by a system of varying forces, and the circumstances of the solidified body will not be the same as those of the fluid, unless its surface is pressed in precisely the same way. These pressures have a total vertical component, which must be added to the weight of the block of ice in order that we may obtain the true pressure on the base.

The action between two portions of a perfect fluid ideally separated by a plane surface of any area always consists of a single force which is normal to the area; but the action between two portions of an elastic solid along a plane section is by no means so simple; the latter is not generally reducible to a single force.

140.] **Equilibrium of several Bodies forming a System.** It will now be clear that when a system is composed of several bodies in contact with each other, we can consider the whole set as forming a single body in equilibrium under the action of given external forces; or we may consider the separate equilibrium of any one body under the action of given external forces, and the reactions of the other bodies with which it is in contact. A few examples of such systems have already been given; but it is proposed to devote the present chapter more especially to the consideration of such questions.

EXAMPLES.

1. Two uniform bars, connected at a common extremity by a smooth joint, are placed in a vertical plane, their other extremities, which rest on a smooth horizontal plane, being connected by a cord; find the tension of the cord and the reaction at the joint.

Let AC and CB (Fig. 167) be the bars, W and W' their weights, α and α' their inclinations to the horizon, R and R' the reactions of the horizontal plane at A and B, and T the tension of the cord.

If, then, we consider the two bars as forming one system, the mutual reaction at C and the tension of the cord will be *internal* forces of the system, and will therefore disappear from the equations of equilibrium. The forces on this system are simply, W, W', R and R'.

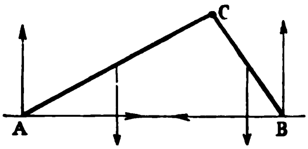

Fig. 167.

Resolving vertically for the equilibrium of the system,

$$R + R' = W + W'. \qquad (1)$$

Again, considering the equilibrium of the bar AC, the forces acting on it are W, R, T, and the unknown reaction at C. This latter will be eliminated by taking moments about C. Thus we get

$$2R\cos\alpha = 2T\sin\alpha + W\cos\alpha,$$

(the length of the bar dividing out), or

$$R = T\tan\alpha + \tfrac{1}{2}W. \qquad (2)$$

Similarly, taking moments about C for the equilibrium of BC,

$$R' = T\tan\alpha' + \tfrac{1}{2}W'. \qquad (3)$$

By adding (2) and (3), and making use of (1), we get

$$T = \frac{W + W'}{2(\tan\alpha + \tan\alpha')}. \qquad (4)$$

Again, let X and Y be the horizontal and vertical components of the reaction at the joint. Then, for the equilibrium of the bar AC,

$$T - X = 0,$$

$$W + Y - R = 0.$$

Hence

$$X = \frac{W + W'}{2(\tan\alpha + \tan\alpha')},$$

$$Y = \frac{W'\tan\alpha - W\tan\alpha'}{2(\tan\alpha + \tan\alpha')}.$$

If we wish to determine T by the principle of virtual work, let y be the height of the middle point of either bar, and we have

$$-(W + W')dy - Td(AB) = 0 \qquad (5)$$

for an imagined displacement in which the bars are drawn out, while A and B remain on the ground. If $AC = 2a$, $BC = 2a'$, $y = a\sin\alpha$, $AB = 2a\cos\alpha + 2a'\cos\alpha'$. Therefore

$$dy = a\cos\alpha d\alpha,\ d(AB) = -2a\sin\alpha d\alpha - 2a'\sin\alpha' d\alpha'$$

$$= -2a\frac{\sin(\alpha + \alpha')}{\cos\alpha'}d\alpha$$

(since from the equation $a\sin\alpha = a'\sin\alpha'$ we have $a\cos\alpha d\alpha = a'\cos\alpha' d\alpha'$).

Substituting these values of dy and $d(AB)$ in (5), we get the same value of T as before.

2. Two equal smooth spheres are placed inside a hollow cylinder, open at both ends, which rests on a horizontal plane; find the least weight of the cylinder in order that it may not be upset.

Let Figure 168 represent a vertical section of the system through the centres of the spheres. Let P be the weight of the cylinder, a its radius, W and r the weight and radius of each sphere, R and R' the reactions between the cylinder and the spheres whose centres are O and O', respectively.

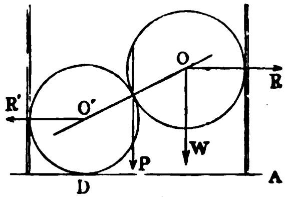

Fig. 168.

Then, the only motion possible for the cylinder is one of tilting over its edge at the point A, in which the vertical plane containing the forces meets it. For, consider the equilibrium of the lower sphere which rests against the ground at D. This sphere is in equilibrium under the influence of R' (reversed in Figure), the reaction of the upper sphere, S, acting in the line OO', its weight, W, and the reaction of the ground at D. Now, since three of these forces pass through O', the reaction of the ground, whether the latter is rough or smooth, must also pass through O'. Hence, if θ be the angle which OO' makes with the horizon, we have for the equilibrium of the lower sphere, resolving horizontally,

$$R' = S\cos\theta. \qquad (1)$$

The upper sphere is in equilibrium under the action of R' (reversed in Figure), W, and S. Hence for its equilibrium we have in the same way,

$$R = S\cos\theta. \tag{2}$$

$$\therefore\ R = R'. \tag{3}$$

Again, the cylinder is in equilibrium under the action of R, R', P, and the reaction of the ground. Resolving horizontally for its equilibrium, we have the horizontal component of the reaction of the ground $= R - R' = 0$. Hence, even if the ground is rough, there is no tendency to slip, and the only way in which equilibrium can be broken is by turning round A.

Taking moments, then, about A, the point at which the reaction of the ground acts, we have for the equilibrium of the cylinder

$$Pa + R'r = R(r + 2r\sin\theta),$$

or

$$Pa = 2Rr\sin\theta. \tag{4}$$

Again, for the equilibrium of the upper sphere, we have

$$\frac{R}{W} = \cot\theta. \tag{5}$$

Substituting this value of R in (4), we have

$$Pa = 2Wr\cos\theta. \tag{6}$$

But evidently $$\cos\theta = \frac{a-r}{r};$$

therefore, finally, $$P = 2W\left(1 - \frac{r}{a}\right).$$

3. A heavy bar is moveable in a vertical plane round a smooth hinge fixed at one extremity; a heavy sphere is attached to the hinge by a cord; the two bodies rest in contact; find the position of equilibrium and the internal reactions, there being no friction between the bodies.

Let O (Fig. 169) be the hinge, OA the cord by which the sphere is attached, θ the inclination of the cord to the vertical, Cm, ϕ the inclination of the bar to the vertical, W the weight and r the radius of the sphere, l the length of the cord, a the distance between O and G, the centre of gravity of the bar, and P its weight.

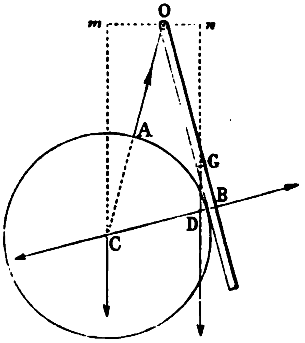

Fig. 169.

Then, considering the sphere and bar as one system, this system is acted on by the given forces W and P, by the tension of the cord, and by the resistance of the hinge. The two latter forces will be eliminated by taking moments about O. We have then

$$W.Om = P.On,$$

Now this equation shows that for a given difference, p, the deflection is less than it would be for the balance if c were made zero; i.e., if O were at C.

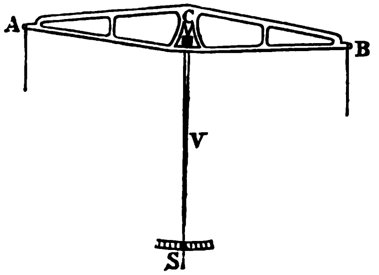

Fig. 163.

Lightness in the beam is attained by making it in the shape of a very elongated lozenge, the central portion of which is nearly all cut out, its sides being, however, braced together by light brass cross-pieces, as represented in Fig. 163, which shows only a few portions of the delicate form of the instrument, as it is used for the very minute weighings in laboratories.

The axis of suspension is, in such an instrument, a knife edge (represented vertically under C) which rests on a fixed horizontal plane of agate. The deflections of the beam from the vertical are indicated by a very long pointer, V, connected with the beam, the end of this pointer moving from right to left just in front of a fixed vertical scale, S, divided into a large number of close divisions.

Absolute equilibrium between the masses in the pans is scarcely ever attained; but even if these masses were equal, some slight disturbance might set the pointer swinging to right and left of its zero position, and if the amplitudes of these swings were equal, we could conclude that the masses in the pans are equal. Hence it is obviously desirable that the *time* of these small oscillations should be as small as possible. The determination of the time of a small oscillation depends on kinetical principles, and we must assume the result here.

If k is the radius of gyration of the beam about the axis of suspension, g the acceleration of a body falling freely in vacuo near the Earth's surface, and if two equal, or very nearly equal, masses, P, are suspended in the pans, the time, t, of a small oscillation is given by the equation

$$t = \pi\sqrt{\frac{Wk^2 + 2Pa^2}{Whg}}; \qquad (2)$$

so that if t is to be very small, i. e., the oscillations rapid, a should be small and h large. These conditions are therefore the reverse of those for sensibility, and nothing but a compromise can be effected between them.

Two methods of *exact weighing*—i. e., weighing to within a milligramme, or even one-tenth of it—are in use, namely the *counterpoise method*, or method of Borda, and the *method of Gauss*.

In the first method a mass of shot or sand (the amount of which we need not know) is put into one pan, while the body to be weighed is put into the other, and with this body are put in known weights (multiples of a gramme, decigramme, etc.) until equilibrium of the pointer is attained either in the middle of the scale S or in some position near it. The body to be weighed is now removed from its pan, and its place is occupied by known masses (grammes, etc.) until the same position of equilibrium (or one near it) of the pointer is again attained. Obviously the mass of the body to be weighed must then be equal to the sum of the masses which have taken its place and which produce the same effect as the body did.

The method of Gauss consists in weighing the body first in one pan and then in the other, and taking the weight of the body equal to the geometric mean of the two (very nearly equal) amounts which seem to represent it. If the arms, OA, OB, of the balance were absolutely equal in length, the body would appear to weigh the same in both pans; but very slight differences between the arms may arise from various causes, and we shall therefore assume the arms to be of different lengths, $OA = a$, $OB = b$. Let M be the true weight of the body. When M is suspended from A, let P_1 be the weight of the known mass which must be suspended from B to bring the pointer to rest in the vertical position; that is, P_1 is the apparent weight of the body, and it appears as if $M = P_1$. The true relation, however, between M and P_1 is

$$M . a = P_1 . b. \tag{3}$$

Again, when M is suspended from B, let P_2 be its apparent weight.

Then

$$M . b = P_2 . a. \tag{4}$$

From these equations we have, by multiplication,

$$M = \sqrt{P_1 . P_2}, \tag{5}$$

which proves the result given above. In practice, since the difference between the apparent weights is extremely minute, it is usual to take

$$M = \tfrac{1}{2} (P_1 + P_2), \tag{6}$$

i. e., the arithmetic, instead of the geometric, mean, the two means being very nearly the same.

For a full description of the practical use and details of the balance the student may very profitably consult the *Lessons in Elementary Practical Physics* by Stewart and Gee, (Vol. I).

135.] **Roberval's Balance.** Roberval's Balance is an excellent illustration of the principle of work.

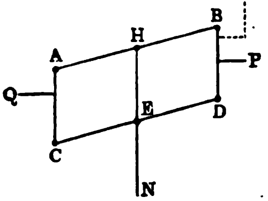

Fig. 164.

Two equal bars, AB and CD (Fig. 164), revolve round axes through their middle points, H and E, which are fixed in a vertical support, HN; these bars are connected by smooth joints to two equal bars, AC and BD, and to these latter bars are rigidly attached two plates or scale pans, P and Q, the points of attachment being any whatever, and one or both of the plates may lie *towards* the vertical support, or away from it (as Fig. 164).

Suppose P and Q to be the weights of two masses placed in the pans P and Q, respectively. Then if for any displacement of the bars round the points H and E, the pans describe vertical spaces p and q, respectively, we shall have for equilibrium

$$Pp - Qq = 0.$$

Now, the bars AC and BD, being always parallel to the fixed line HE, will be always vertical, and the vertical space through which one moves up is obviously equal to that through which the other moves down. Hence $p = q$, and we have for equilibrium

$$P = Q,$$

whatever be the lengths of the pans (provided their weights are neglected), whatever be their points of attachment to BD and AC, and whatever the points in the pans at which P and Q are placed.

If the weights of the pans are taken into account, the same results follow if they are of equal weight.

If the pan P were replaced by the pan P', and the weight P placed at P', the other pan, Q, remaining unchanged, and the weights of the pans being either equal or neglected, equilibrium would still subsist—a result which seems at first sight very strange.

If the lengths AH and HB, CE and ED are not equal, it is easy to prove that $\frac{p}{q} = \frac{HB}{HA}$, and the condition of equilibrium is

$$P.HB = Q.HA.$$

136.] **Balance of Quintenz.** This is a compound balance formed of a combination of several levers, and is used for weighing very heavy loads. This machine also furnishes an admirable example of the principle of work.

AB (Fig. 165) is a lever moveable about its fixed extremity, *A*; *MN* is another lever moveable about a fulcrum, *F*, fixed at its middle point; *CD* is a moveable platform, which receives the load *Q*, whose weight is to be found; this platform is connected with the lever *MN* by a rigid vertical bar, *DI*, articulated at *D* and *I*; and the platform further rests against the lever, *AB*, by an edge of contact at a fixed point, *H*, on the latter; finally, the two levers are connected by a rigid vertical bar, *BM*, articulated to both.

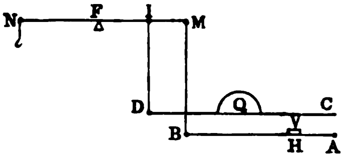

Fig. 165.

The weight, *P*, employed to measure *Q* is applied to the upper lever at *N*. Let the system receive any slight displacement, then the lever, *AB*, will turn round *A* through an angle $\delta\theta$, suppose, and the lever *MN* will turn round *F* through an angle $\delta\phi$.

We shall arrange the dimensions of the machine in such a manner that the platform, *CD*, may remain horizontal in the displacement. The vertical descent of the point *H* is evidently $AH.\delta\theta$, and this is also the vertical descent of the point in the platform above *H*.

The vertical descent of the point *D* is the same as that of *I*, and this latter is obviously $FI.\delta\phi$; hence if the platform remains horizontal, $FI.\delta\phi = AH.\delta\theta$.

Again, the vertical descent of *M* is the same as that of *B*; or

$$FM.\delta\phi = AB.\delta\theta.$$

Hence from these equations we have

$$\frac{MF}{FI} = \frac{BA}{AH},$$

which is the condition for the horizontality of the platform. Denote $\frac{BA}{AH}$ by n. The equation of work is obviously

$$P \times \text{descent of } N = Q \times \text{descent of } D,$$

$$\therefore \quad P \, . \, NF \, \delta\phi = Q \, . \, FI \, \delta\phi,$$

$$\therefore \quad P = \frac{1}{n} Q,$$

or the result is the same as if Q were suspended from the point I of the upper lever.

Loads placed on the platform may all be weighed by means of a constant weight, P, by merely moving the point of suspension of this latter along the arm NF; thus, if P is suspended from the point K between N and F, we shall have

$$\frac{P}{Q} = \frac{FI}{FK}.$$

137.] **Toothed Wheels.** Motion may be transferred from one point to another and work done by means of a combination of toothed wheels, each one of which drives the next one in the series. The discussion of this kind of machinery possesses great geometrical elegance; but the space at our disposal renders it impossible to do more than give a slight sketch of the simplest case—that in which the axes of the wheels are all parallel.

For the investigation of the proper forms of teeth, the student is referred to Willis's *Principles of Mechanism*, Collignon's *Statique*, and Resal's *Mécanique Générale*.

Fig. 166 represents a toothed wheel, A_1, moveable round a horizontal axis, ab; the effort, P, is applied by means of a handle, cd, which, when turned, causes the axis ab to rotate in its bearings at a and b and to turn the wheel A_1; this wheel causes another, B_1, in contact with it, to rotate round a horizontal axis which also moves in fixed bearings at its extremities; on this latter axis is fixed another wheel A_2, whose rotation in like manner turns B_2 on its axis, which in the figure is the axis of a cylinder to which the resistance, Q, is attached.

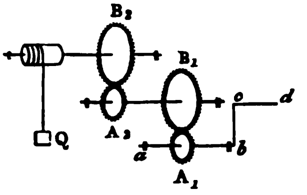

Fig. 166.

Suppose that there are n wheels, A_1, A_2, ... A_n, whose radii are $a_1, a_2, \ldots a_n$, and n wheels, $B_1, B_2, \ldots B_n$, whose radii are $b_1, b_2, \ldots b_n$; and let $bc = p$, and the radius of the cylinder (or wheel) to which Q is attached $= q$. Then, if ω_1 is the angle through which the radius bc revolves, the effort being always applied tangentially to the circle described by its point of application, the work expended is

$$Pp\omega_1;$$

and if ω_n is the angle through which, in the same time, the cylinder rotates, the weight Q will be raised through a distance $q\omega_n$, and the work done against the resistance is

$$Qq\omega_n.$$

Supposing then that no work is lost either by the friction of the axes in their bearings, or by the friction of the teeth against each other, we must have

$$Pp\omega_1 = Qq\omega_n,$$

when the machine is moving uniformly.

To determine the kinematical relation between ω_1 and ω_n, let the angle through which B_1 turns be ω_2. Then, since the distances described by the points of A_1 and B_1 which are in contact are the same, $a_1\omega_1 = b_1\omega_2$. Also if ω_3 is the angle through which B_2 turns, we have $a_2\omega_2 = b_2\omega_3$. Proceeding in this way, we have, by multiplying the corresponding sides of these equations together, $a_1 a_2 \ldots a_n \,.\, \omega_1 = b_1 b_2 \ldots b_n \,.\, \omega_n$.

Hence from (1) and (2),

$$\frac{Q}{P} = \frac{p}{q} \cdot \frac{b_1 b_2 \ldots b_n}{a_1 a_2 \ldots a_n}.$$

For the calculation of the work lost by the friction of the teeth among themselves see Collignon's *Statique*, p. 468.

CHAPTER IX.

DETERMINATION OF MUTUAL REACTIONS OF PARTS OF A SYSTEM.

138.] **Action and Reaction.** If in any system of bodies, connected in any manner, A and B are two bodies in contact between which an action of some kind is exercised; then, whatever be the forces with which the body A acts upon the body B, the very same forces, reversed in directions, will constitute the action of B on A. Let the whole system of forces acting on A, excluding those produced by B, be denoted by (P), and let the forces constituting the action of B on A be denoted by (R); then we may sever the connexion between A and B, provided that we have other means of producing on A the system of forces (R). In the same way, if (Q) denote the whole system of forces acting on B, those constituting the action of A on it being excluded, the body, B, may be severed from A provided that we have the means of producing a system of forces $(-R)$ on B, $(-R)$ denoting a system of forces obtained by reversing the direction and preserving the magnitude of every force in (R).

For example, the beam CD (Fig. 126) may be severed from the other beam along any section, CB, provided that there be introduced on CD either the single force R acting through A, or the complex system of tensile and compressive forces which act at the section CB. This equality of magnitude and oppositeness of direction of the forces existing between two distinct bodies in contact, or between ideally severed portions of the same body, is sometimes spoken of as the principle of *the equality of Action and Reaction*; but this is by no means the whole of the Newtonian principle called by this name; for Newton specifies several senses in which the terms Action and Reaction can be taken, and in discussing one of them he has explicitly anticipated, in great part, the principle of the Conservation of Energy—as has been pointed out by Thomson and Tait.

139.] **Examples of Internal Action.** The cases which we shall consider in this chapter are those in which the action between two portions of a system ideally severed consists of a single force. The simplest example of such action occurs when a single point of one body rests against the surface of another, the bodies being either rough or smooth. If the bodies are smooth, the action between them consists of a single force which is *normal to the surface of contact*; and if rough, the action is still a single force which is not necessarily normal to this surface. In all cases in which smooth spherical joints or hinges are concerned, the action exercised on bodies connected by them consists of a single force passing through the centre of the joint. When rough joints are used, the action will generally consist of a single force acting somewhere outside the joint; or of a couple and a force acting at the joint; or, possibly, of a couple alone. The tension of a string is also an instance of internal action, and its nature has been already explained in Chap. II.

Again, if we ideally separate into two portions by an arbitrary surface a mass of a perfect fluid in equilibrium, the action of one portion on the other over a small area of the ideally separating surface will consist of a single force acting normally on the area. And we may always treat as a separate body any portion whatever of a fluid in equilibrium*, *provided that we produce along the surface of this ideally separated portion all the forces which are actually produced on it by the fluid with which it was surrounded.* This is what we have already enunciated as *the principle of separate equilibrium* (p. 32). It is by such separate consideration of portions of a fluid that we arrive at a knowledge of its internal forces or pressures. For example, if a heavy fluid, whether compressible or incompressible, of uniform or varying density, be contained in a vessel, we can prove that the pressure is the same at all points, P, Q, in the same horizontal plane. For, isolate in imagination a horizontal cylindrical column of the fluid, having small vertical and equal areas at P and Q for extremities, from the rest of the fluid. Then, we may treat the cylinder of fluid PQ as a separate body, provided that, in addition to the external force (gravity) acting on it, we introduce the forces which it actually experienced from the surrounding fluid. Now these forces consist of normal pressures, p and q, on the areas at P and Q, together with normal pressures all over its curved surface, these latter being all at right angles to the axis PQ. If now we resolve horizontally all the forces acting on the cylinder, we get

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* It is usually said that we may, under the above condition, imagine any portion of the fluid *to become solidified*; but the imagined *solidification* is not only wholly unnecessary but misleading.

$$p-q=0, \text{ or } p=q.$$

This demonstration shows, moreover, that in the case of a heavy viscous or imperfect fluid, the pressures are not necessarily equal at all points in the same horizontal plane.

For, in this case, the action of the rest of the fluid on PQ does not necessarily consist of forces normal to its surface, but of oblique forces. Hence the horizontal component of the pressure at P is not equal to the horizontal component at Q; the difference between them is equal to the sum of the horizontal components of the oblique forces.

The importance of keeping such considerations in view may be illustrated by the following example from Hydrostatics.

A conical vessel is filled with water through an aperture at the vertex. From Hydrostatical principles it follows that the pressure on the base of the cone is equal to the weight of a *cylindrical* column of water, standing on the base, and having a height equal to that of the cone; that is, the pressure on the base is much greater than the weight of the water contained in the cone. Now if we imagine the water to become solidified, the curved surface of the cone may be removed, and the pressure on the base will be equal to the weight of the ice, that is, the weight of the water in the cone. An apparent discrepancy is the result. But if we attend to the proviso that in the separate consideration of the equilibrium of any portion of a system, solid or fluid, we must produce upon the isolated portion all the forces which were originally produced upon it by the neighbouring portions of the solid or fluid, the difficulty disappears. In the fluid state the liquid in contact with the curved surface of the cone was pressed normally by a system of varying forces, and the circumstances of the solidified body will not be the same as those of the fluid, unless its surface is pressed in precisely the same way. These pressures have a total vertical component, which must be added to the weight of the block of ice in order that we may obtain the true pressure on the base.

The action between two portions of a perfect fluid ideally separated by a plane surface of any area always consists of a single force which is normal to the area; but the action between two portions of an elastic solid along a plane section is by no means so simple; the latter is not generally reducible to a single force.

140.] **Equilibrium of several Bodies forming a System.** It will now be clear that when a system is composed of several bodies in contact with each other, we can consider the whole set as forming a single body in equilibrium under the action of given external forces; or we may consider the separate equilibrium of any one body under the action of given external forces, and the reactions of the other bodies with which it is in contact. A few examples of such systems have already been given; but it is proposed to devote the present chapter more especially to the consideration of such questions.

EXAMPLES.

1. Two uniform bars, connected at a common extremity by a smooth joint, are placed in a vertical plane, their other extremities, which rest on a smooth horizontal plane, being connected by a cord; find the tension of the cord and the reaction at the joint.

Let AC and CB (Fig. 167) be the bars, W and W' their weights, a and a' their inclinations to the horizon, R and R' the reactions of the horizontal plane at A and B, and T the tension of the cord.

If, then, we consider the two bars as forming one system, the mutual reaction at C and the tension of the cord will be *internal* forces of the system, and will therefore disappear from the equations of equilibrium. The forces on this system are simply, W, W', R and R'.

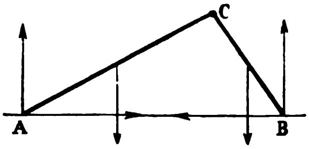

Fig. 167.

Resolving vertically for the equilibrium of the system,

$$R+R'=W+W'. \qquad (1)$$

Again, considering the equilibrium of the bar AC, the forces acting on it are W, R, T, and the unknown reaction at C. This latter will be eliminated by taking moments about C. Thus we get

$$2R\cos a = 2T\sin a + W\cos a,$$

(the length of the bar dividing out), or

$$R = T\tan a + \tfrac{1}{2}W. \qquad (2)$$

Similarly, taking moments about C for the equilibrium of BC,

$$R' = T\tan a' + \tfrac{1}{2}W'. \qquad (3)$$

By adding (2) and (3), and making use of (1), we get

$$T = \frac{W + W'}{2(\tan\alpha + \tan\alpha')}. \tag{4}$$

Again, let X and Y be the horizontal and vertical components of the reaction at the joint. Then, for the equilibrium of the bar AC,

$$T - X = 0,$$

$$W + Y - R = 0.$$

Hence

$$X = \frac{W + W'}{2(\tan\alpha + \tan\alpha')},$$

$$Y = \frac{W'\tan\alpha - W\tan\alpha'}{2(\tan\alpha + \tan\alpha')}.$$

If we wish to determine T by the principle of virtual work, let y be the height of the middle point of either bar, and we have

$$-(W + W')dy - Td(AB) = 0 \tag{5}$$

for an imagined displacement in which the bars are drawn out, while A and B remain on the ground. If $AC = 2a$, $BC = 2a'$, $y = a\sin\alpha$, $AB = 2a\cos\alpha + 2a'\cos\alpha'$. Therefore

$$dy = a\cos\alpha\, d\alpha,\quad d(AB) = -2a\sin\alpha\, d\alpha - 2a'\sin\alpha'\, d\alpha'$$

$$= -2a\frac{\sin(\alpha + \alpha')}{\cos\alpha'}d\alpha$$

(since from the equation $a\sin\alpha = a'\sin\alpha'$ we have $a\cos\alpha\, d\alpha = a'\cos\alpha'\, d\alpha'$).

Substituting these values of dy and $d(AB)$ in (5), we get the same value of T as before.

2. Two equal smooth spheres are placed inside a hollow cylinder, open at both ends, which rests on a horizontal plane; find the least weight of the cylinder in order that it may not be upset.

Let Figure 168 represent a vertical section of the system through the centres of the spheres. Let P be the weight of the cylinder, a its radius, W and r the weight and radius of each sphere, R and R' the reactions between the cylinder and the spheres whose centres are O and O', respectively.

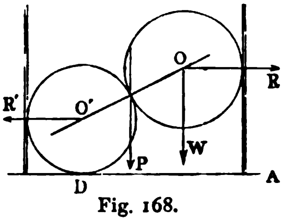

Fig. 168.

Then, the only motion possible for the cylinder is one of tilting over its edge at the point A, in which the vertical plane containing the forces meets it. For, consider the equilibrium of the lower sphere which rests against the ground at D. This sphere is in equilibrium under the influence of R' (reversed in Figure), the reaction of the upper sphere, S, acting in the line OO', its weight, W, and the reaction of the ground at D. Now, since three of these forces pass through O', the reaction of the ground, whether the latter is rough or smooth, must also pass through O'. Hence, if θ be the angle which OO' makes with the horizon, we have for the equilibrium of the lower sphere, resolving horizontally,

$$R' = S\cos\theta. \tag{1}$$

The upper sphere is in equilibrium under the action of R' (reversed in Figure), W, and S. Hence for its equilibrium we have in the same way,

$$R = S\cos\theta. \qquad (2)$$

$$\therefore\ R = R'. \qquad (3)$$

Again, the cylinder is in equilibrium under the action of R, R', P, and the reaction of the ground. Resolving horizontally for its equilibrium, we have the horizontal component of the reaction of the ground $= R - R' = 0$. Hence, even if the ground is rough, there is no tendency to slip, and the only way in which equilibrium can be broken is by turning round A.

Taking moments, then, about A, the point at which the reaction of the ground acts, we have for the equilibrium of the cylinder

$$Pa + R'r = R(r + 2r\sin\theta),$$

or

$$Pa = 2Rr\sin\theta. \qquad (4)$$

Again, for the equilibrium of the upper sphere, we have

$$\frac{R}{W} = \cot\theta. \qquad (5)$$

Substituting this value of R in (4), we have

$$Pa = 2Wr\cos\theta. \qquad (6)$$

But evidently $\cos\theta = \dfrac{a-r}{r}$;

therefore, finally, $P = 2W\left(1 - \dfrac{r}{a}\right)$.

3. A heavy bar is moveable in a vertical plane round a smooth hinge fixed at one extremity; a heavy sphere is attached to the hinge by a cord; the two bodies rest in contact; find the position of equilibrium and the internal reactions, there being no friction between the bodies.

Let O (Fig. 169) be the hinge, OA the cord by which the sphere is attached, θ the inclination of the cord to the vertical, Cm, ϕ the inclination of the bar to the vertical, W the weight and r the radius of the sphere, l the length of the cord, a the distance between O and G, the centre of gravity of the bar, and P its weight.

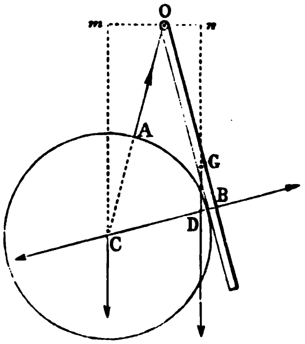

Fig. 169.

Then, considering the sphere and bar as one system, this system is acted on by the given forces W and P, by the tension of the cord, and by the resistance of the hinge. The two latter forces will be eliminated by taking moments about O. We have then

$$W.Om = P.On,$$

Om and *On* being perpendiculars from O on the directions of W and P. But $Om = (l+r)\sin\theta$, and $On = a\sin\phi$; therefore

$$W.(l+r)\sin\theta = P.a\sin\phi. \qquad (1)$$

This is the statical equation connecting θ and ϕ; the geometrical equation is

$$\sin COB = \frac{r}{l+r}, \quad \text{or}$$

$$\sin(\theta+\phi) = \frac{r}{l+r}. \qquad (2)$$

(1) and (2) determine θ and ϕ, and therefore the position of equilibrium. If R is the mutual reaction of the sphere and the bar, we have, by considering the equilibrium of the sphere alone,

$$R = W\frac{\sin\theta}{\cos(\theta+\phi)}. \qquad (3)$$

Again, if the cord is attached to the hinge but not to the bar, and if X and Y are the horizontal and vertical components of the pressure of the bar on the hinge, we have for the equilibrium of the bar

$$X = R\cos\phi = W\frac{\sin\theta\cos\phi}{\cos(\theta+\phi)},$$

$$Y = P - R\sin\phi = P - W\frac{\sin\theta\sin\phi}{\cos(\theta+\phi)}.$$

Hence, if S is the resultant of X and Y,

$$S^2 = P^2 - 2PW\frac{\sin\theta\sin\phi}{\cos(\theta+\phi)} + W^2\frac{\sin^2\theta}{\cos^2(\theta+\phi)}. \qquad (4)$$

Evidently S acts in the line OD, which joins the hinge to the point of intersection of P and R.

If the cord is attached to the bar, X and Y are the components of the resultant of the tension of the cord and the pressure on the hinge.

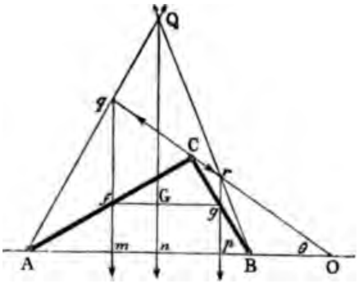

Fig. 170.

4. Two heavy uniform rods are freely jointed at a common extremity, and are connected at their other extremities with two smooth hinges in the same horizontal line. Required the magnitudes and directions of the pressures on the hinges, and the mutual reaction between the rods.

Let AC and CB (Fig. 170) be the rods; W and W' their weights, acting through their middle points, f and g; a and a' their inclinations to the horizon; R the mutual reaction at C; S and S' the pressures on the hinges A and B, G the centre of gravity of the system of two rods; and θ the inclination of R to the horizon.

Consider the equilibrium of AC alone. It is acted on by three forces W, R, and S; and since we have drawn the line OC to represent the direction of R, the direction of S must be Aq, q being the point of intersection of W and R. By taking moments about A for the equilibrium of AC, we shall express R in terms of W, α, and θ; and by taking moments about B for the equilibrium of BC, we shall express R in terms of W', α', and θ; equating the two values of R thus obtained, we get a value for $\tan\theta$ which is obtained by dividing the value of Y by that of X in Example 1.

Considering the two rods as one system, this system is acted on by the three external forces, S, S', and $W+W'$, acting vertically through G. Hence these must meet in a point, Q.

It is evident that this problem is the same as that in Example 1, and that if the reactions S and S' are resolved each into a vertical and a horizontal component, the horizontal components will be equal and opposite (by considering the two rods as one body and resolving horizontally). These horizontal components have each the value of the tension of the cord in Example 1, and the vertical components are the values of R and R'. Thus the problem might be completely solved analytically.

*Geometrical Solution**. The direction of the resistance at the joint C can be easily determined as follows:—From A and B draw two lines to any point, D, on the line QG; let AD meet qf in E, and let BD meet rg in H. Then the line EH will meet AB in O, the point through which the line of resistance at C passes. For, the triangles qrQ and EHD are such that the lines, Eq, DQ, Hr, joining corresponding vertices meet in a point (are parallel), therefore, by the well-known property of triangles in perspective (which has been given at p. 141), the intersections, A, B, O, of corresponding sides must lie on a right line. Hence O is determined, and therefore OC, the line of resistance.

The direction of R can also be found thus geometrically:—

Since qrO is a transversal cutting the sides of a triangle AQB, we have

$$\frac{AO}{OB}=\frac{Aq}{qQ}\times\frac{Qr}{rB}=\frac{Am}{mn}\times\frac{np}{pB}=\frac{Am}{pB}\times\frac{np}{mn}$$

$$=\frac{Am}{pB}\times\frac{gG}{fG}=\frac{AC\cos\alpha}{BC\cos\alpha'}\cdot\frac{W}{W'}.$$

But $AO=AC\dfrac{\sin(\alpha+\theta)}{\sin\theta}$, and $OB=BC\dfrac{\sin(\alpha'-\theta)}{\sin\theta}$; therefore

$$\frac{\sin(\alpha+\theta)}{\sin(\alpha'-\theta)}=\frac{\cos\alpha}{\cos\alpha'}\cdot\frac{W}{W'},$$

from which we get the same value of $\tan\theta$ as before.

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* This elegant solution was suggested to me by Mr. Henry Reilly.

5. A sphere and a cone, each resting on a smooth inclined plane, are placed in contact; find the position of equilibrium of the system, and the reactions of the planes.

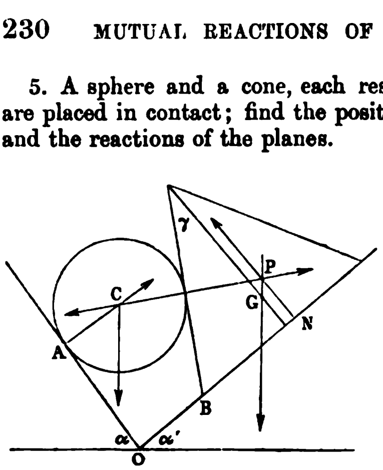

Fig. 171.

Let the sphere rest on the plane OA (Fig. 171) whose inclination to the horizon is α, and the cone on OB whose inclination is α'; let W and W' be the weights of the sphere and cone, R the mutual reaction between them, S the reaction of the plane OA on the sphere, T the reaction of OB on the cone, and let γ be the semivertical angle of the cone.

For the equilibrium of the sphere we have

$$R = W\frac{\sin\alpha}{\cos(\alpha+\alpha'-\gamma)}; \qquad (1)$$

and for the equilibrium of the cone

$$R = W'\frac{\sin\alpha'}{\cos\gamma}. \qquad (2)$$

From (1) and (2) we have

$$W\frac{\sin\alpha}{\cos(\alpha+\alpha'-\gamma)} = W'\frac{\sin\alpha'}{\cos\gamma}, \qquad (3)$$

an equation which, instead of giving a position of equilibrium, gives *a condition to be satisfied* in order that equilibrium may be at all possible.

It is evident that (3) is the only statical equation that can be obtained without involving the unknown reactions. Hence, if it is satisfied, every position in which the bodies are placed is one of equilibrium; and if it is not satisfied, the problem must be radically changed, and one or other of the two bodies must rest in contact with *both* planes. Suppose the cone in contact with both planes.

Here there are only three forces acting on the sphere, and there are four forces acting on the cone, viz. W', R, T, and F, the reaction of the plane OA, which is perpendicular to OA. R must now be determined from the equilibrium of the sphere. Thus

$$R = W\frac{\sin\alpha}{\cos(\alpha+\alpha'-\gamma)}.$$

To determine F, consider the equilibrium of the cone, and resolve along OB. Then

$$F = \left[W'\sin\alpha' - W\frac{\sin\alpha\cos\gamma}{\cos(\alpha+\alpha'-\gamma)}\right]\operatorname{cosec}(\alpha+\alpha').$$

To determine the magnitude of *T*, resolve the forces on the cone in the direction *OA*. Then

$$T = (W + W')\frac{\sin a}{\sin(a + a')}.$$

The point *N* at which *T* acts is obtained by taking moments about *O* for the equilibrium of the cone. We thus get

$$T.ON = W'h\left(\tan\gamma\cos a' - \frac{1}{4}\sin a'\right) + Rr\cot\left(\frac{\pi}{4} - \frac{a + a' - \gamma}{2}\right),$$

r being the radius of the sphere, and *h* the height of the cone.

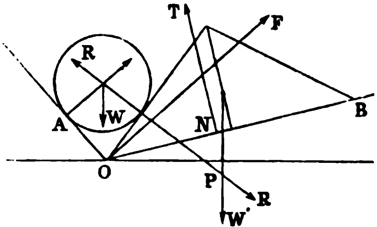

Fig. 172.

ON is obtained by substituting in this equation the values of *T* and *R* given above, and it is geometrically evident that the point *N* lies between the foot of the perpendicular from *P* on *OB* and the foot of the perpendicular from the intersection of *F* and *W'* on *OB*.

If the sphere is in contact with both planes, the discussion proceeds in a similar manner. *R* is then determined from the equilibrium of the cone, *T* acts in the perpendicular from *P* on *OB*, and the reactions of the planes on the sphere are easily calculated.

If the weight of the sphere be greater than the value

$$W'\frac{\sin a' . \cos(a + a' - \gamma)}{\sin a\cos\gamma}$$

given by (3), it is sufficiently clear that the sphere will descend to contact with the plane *OB*; whereas if it is less than this value, the cone will descend.

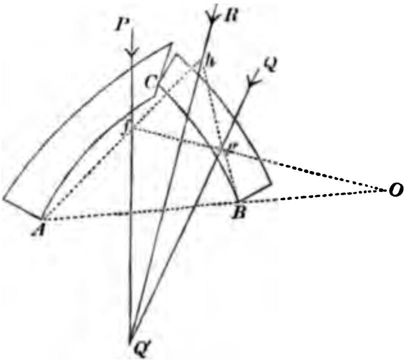

Fig. 173.

If the condition (3) is satisfied, the reaction *T* of the plane *OB* on the cone is easily found. For, let the directions of *W'* and *R* meet in *P*; then *T* must act in the perpendicular, *PQ*, from *P* on *OB*, and

$$T = W' . \frac{\cos(a' - \gamma)}{\cos\gamma}.$$

Similarly *S* may be found.

6. Two blocks, *AC* and *BC* (Fig. 173), rest against two fixed supports at *A* and *B*, and against each other at *C*; each is acted on by

a given force (in addition to its weight); find the lines of resistance at A, B, C.

Ans. Let the resultant of the weight of the block AC and the force applied to it be the force P; let the resultant of the weight of BC and the force applied to it be Q; and let the resultant of P and Q be R. Draw the line AB; take any point, h, on R, and draw Ah and Bh, meeting P and Q in f and g, respectively. Then the line fg will intersect AB in O, the point through which the line of resistance at C passes. Draw OC, and let it meet P in F and Q in G. Then AF and BG are the lines of resistance at A and B. (See Example 4.)

7. A number n of equal smooth spheres of weight W and radius r are placed within a hollow vertical cylinder of radius a, less than $2r$, open at both ends and resting on a horizontal plane. Prove that the least value, W', of the weight of the cylinder in order that it may not be upset by the spheres is given by

$$aW' = (n-1)(a-r)W,$$

or

$$aW' = n(a-r)W,$$

according as n is odd or even. (*Tripos*, 1884.)

141.] **System of Jointed Bars.** When a system consists of a number of rods or bars articulated, or connected together by smooth joints, there will be exerted at the extremities of each rod certain forces, or *reactions*, which are produced by the connecting joints, and the calculation of the directions and magnitudes of these reactions forms an important part of Statics as applied to the construction of framework. These reactions are sometimes called *stresses*, although this term is more properly applied to forces produced by strains or deformations in a body.

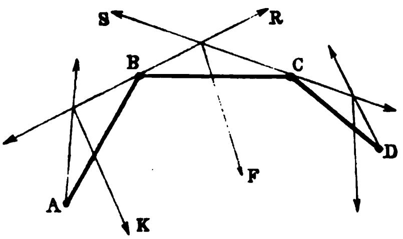

Fig. 174.

The joint connecting any two bars may be either a portion of one of the bars or a hinge-pin distinct from both bars, and the directions of the reactions at the extremities of a bar will depend on the manner in which the external forces are applied. Let us suppose that the joints at B and C (Fig. 174), which connect the bar BC with the neighbouring bars, are distinct from BC itself, and that the forces applied to the system act at and *on* the joints. Then the reactions produced at B and C on the bar *BC* act along this bar. For, the only forces[*] acting on the bar are the reactions of the joints *B* and *C*, and when two forces keep a body in equilibrium, they must be equal and opposite. Hence the reactions must act along *BC*. Suppose, however, that the forces, still applied at the joints, act on the extremities of the bar *BC* itself, and let Fig. 175 represent the bar apart from the joints. Let the forces applied to it be *P* and *Q*. Now the smooth joints must produce reactions which act on the bar through the centres of the joints (see p. 159). Hence *BC* is again kept in equilibrium by forces acting at its extremities, and therefore the resultant of the forces at *B* must be a force acting in the direction *BC* or *CB*, and the resultant of the forces at *C* must be a force acting in the direction *CB* or *BC*. Hence the reactions produced by the joints cannot act along the bar, but must assume some such directions as *R* and *S*.

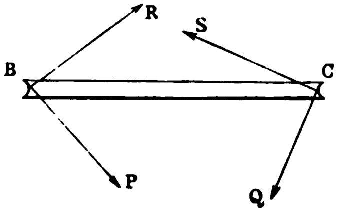

Fig. 175.

Thus, *in any system of articulated bars, when the external forces are applied at the joints, the reactions will be in the directions of the bars only when the external forces act at the joints on pins which are distinct from the bars which they connect.*

142.] **Theorem.** When a system of articulated bars is in equilibrium under the action of external forces applied at given points in the bars, the statical condition of the system may be determined by resolving the force applied to each bar into any two components acting on the joints at its extremities, and then representing each joint as in equilibrium under the action of the components transferred to it together with reactions acting on it along the directions of the bars which it connects.

Let Fig. 176 represent one of the bars detached from the joints at its extremities, and let Fig. 177 represent the joint which connected the bars *AB* and *BC* (Fig. 174). If a force *F* is applied to *BC*, it is, of course, allowable to break it up into any two components, *P* and *Q*, acting on the bar. Let *P* and *Q* act on the bar at its extremities, and let *R* be the reaction of the joint at *B* on the bar, and *S* that of the joint at *C*. The bar is then kept in equilibrium by the forces P and R at B, and the forces Q and S at C. Hence the resultant of P and R must be a force, T, along the bar; that is to say, if the forces P and R act at any point, they produce a resultant T; or again, if we reverse the directions of R and T (as in Fig. 177), the forces P and T are equivalent to R. Now the joint was kept in equilibrium by the equal and opposite reactions, R and R' (Fig. 177), of the bars BC and AB. But we have just shown that R is equivalent to the transferred component P of the force F and the reaction T, acting along CB. In the same way, R' may be replaced by a component of the force K (Fig. 174) acting on AB and a reaction acting along AB.

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[*] The weight of the bar is supposed to be neglected.

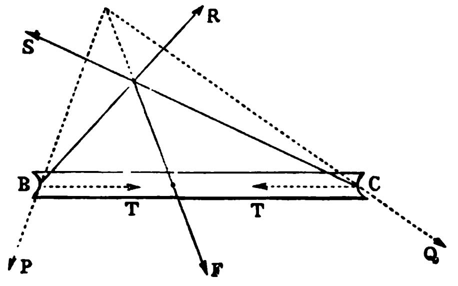

Fig. 176.

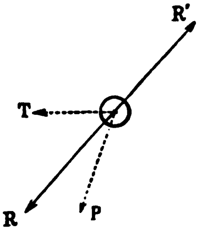

Fig. 177.

We may, then, replace the external forces, K, F, ... (Fig. 174), which act on the bars by any system of components passing through the centres of the joints, and represent two equal and opposite reactions as acting at the extremities and in the direction of each bar of the system. But it must be remembered that the reactions thus calculated (such as T, Fig. 176) are not the *total* reactions at the joints.

The reaction at the end of each bar, thus calculated, is the resultant of the total reaction at the joint and the component of the force acting on the bar which has been transferred to the joint.

For example, the reaction along the bar AB is the resultant of the total reaction, R, and the component of K which has been transferred to the joint B.

The external forces, F, K, ... may be each broken up into two components passing through the centres of the corresponding joints in an infinite number of ways. In the calculation of reactions in framework it is usual to break each of them up into two parallel forces.

143.] **Triangular Framework, Graphic Calculation.** Let ABC (Fig. 178) be a triangular system of bars connected by smooth joints; and let given forces, P, Q, R, keeping the system in equilibrium, be applied at given points to the bars BC, CA, AB, respectively. It is required to find the reactions at the joints.

The reactions on BC at B and C must, for the equilibrium of this bar, meet on P. Suppose that they act along aC and aB. Similarly the reactions on AB at A and B must meet on R.

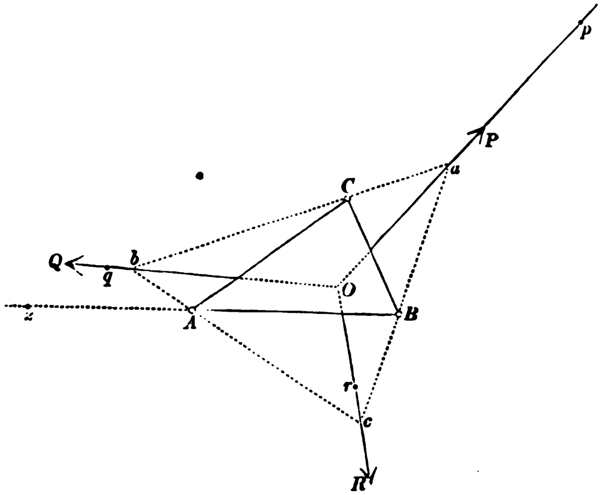

Fig. 178.

Suppose that they act in cA and cB. And let the reactions on AC act in bA and bC. Then aBc is a right line, since 'action and reaction' at B are equal and opposite. Similarly cAb and bCa are right lines. Hence abc is a triangle whose sides pass through three given points, A, B, C, and whose vertices lie on three concurrent lines, P, Q, R. This triangle is therefore found (p. 141) by taking any point, r, on R; let rA meet Q in q, and let rB meet P in p; let pq meet AB in z; join z to C; then zC meets P and Q in a and b, while bA and aB meet R in c. We have thus found the triangle abc, along whose sides the reactions act.

Let T_1, T_2, T_3, be the magnitudes of the reactions at A, B, C.

Then $\dfrac{T_2}{T_3} = \dfrac{\sin CaO}{\sin BaO} = \dfrac{Ob \, . \sin aOb}{Oc \, . \sin cOa} \cdot \dfrac{ac}{ab} = \dfrac{ac \, . \, Ob \, . \, R}{ab \, . \, Oc \, . \, Q}$.

Hence $T_1 : T_2 : T_3 = \dfrac{bc \, . \, Oa}{P} = \dfrac{ca \, . \, Ob}{Q} = \dfrac{ab \, . \, Oc}{R}$.

If the perpendiculars of the triangle abc drawn from the vertices are p_1, p_2, p_3, the actual magnitude of $T_1 = \dfrac{Oa}{p_1} \cdot \dfrac{QR \sin bOc}{P}$,

or $\dfrac{2\,Oa}{p_1} \cdot \dfrac{\sqrt{s \, . \, s-P \, . \, s-Q \, . \, s-R}}{P}$; where $s \equiv \frac{1}{2}(P+Q+R)$;

with similar values of T_2 and T_3.

The triangle abc may be regarded as a funicular polygon of the given forces P, Q, R.

144.] **Deformable Polygon of Bars.** Let a plane polygon of n sides be formed by n bars *rigidly* joined together at their extremities, and let n forces, P_1, P_2, ... P_n, in the plane of the polygon, be applied, one to each bar at a given point in its length.

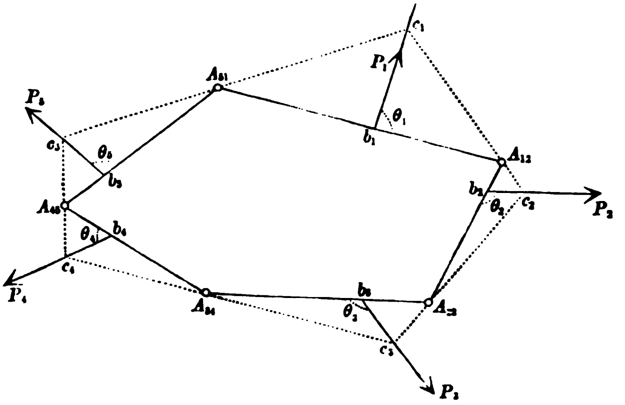

Fig. 179.

Then if the force and funicular polygons of the given forces are both closed, the figure is in equilibrium. Now let the rigidity be removed from the joints, and let them become perfectly free. The system will no longer, in general, remain in equilibrium, because of the restriction now imposed on the internal force between bar and bar—viz., that it must act through their point of junction (see Art. 106). Let us suppose the polygon to remain in equilibrium and investigate the condition for this.

Fig. 179 represents a polygon of five bars acted upon by forces P_1, P_2, ... P_5 whose force and funicular polygons are closed.

Consider the separate equilibrium of the bar $A_{51}A_{12}$. It is acted on by three forces, viz., P_1 and the reactions at its extremities. These must meet in a point, c_1. Similarly the reactions at A_{51} and A_{45} must meet in a point, c_5, on P_5; and the equilibrium of the joint A_{51} requires that $c_1 A_{51} c_5$ should be a right line, and that the components of P_1 and P_5 along it (the other components of these forces being along $c_1 A_{12}$ and $c_5 A_{45}$, respectively) should be equal and opposite. Producing $c_1 A_{12}$ to meet P_2 in C_2, and so on all round, we obtain a polygon $c_1 c_2 c_3 c_4 c_5 c_1$ which is a funicular of the given forces. Hence the necessary and sufficient condition of equilibrium of the deformable polygon is that—*It is possible to describe a funicular polygon of the given forces whose sides all pass, in order, through the joints of the deformable polygon of bars.*

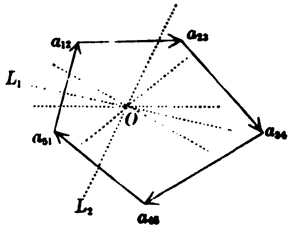

Fig. 180.

Analytical expression may be given to this condition by finding the locus of the pole of a funicular polygon two of whose sides pass through A_{12} and A_{23}, the locus of the pole of a funicular two of whose sides pass through A_{23} and A_{34}, and so on; and expressing that all these loci intersect in a common point, O, which is the pole of the funicular which circumscribes the polygon of bars. By Art. 96 it is obvious that the locus of the pole of a funicular which passes through A_{51} and A_{12} is a right line, OL_1 (Fig. 180), parallel to the bar $A_{51}A_{12}$; and this may otherwise be seen by the *cotangent formula* of Art. 36, thus: denote the length of this bar (to which the force P_1 is applied) by $l^{(1)}$, and denote the segments, $b_1 A_{12}$ and $b_1 A_{51}$, into which P_1 divides it by $l_2^{(1)}$ and $l_5^{(1)}$, the first being that adjacent to the vertex A_{12} and the second adjacent to A_{51}. Then if O is any position of the pole, draw OL_1 parallel to $A_{12}A_{51}$, and we easily see that OL_1 is a fixed line; for, in the triangle $c_1 A_{51} A_{12}$ we have

$$l_1 \cot \theta_1 = l_2^{(1)} \cot a - l_5^{(1)} \cot \beta,$$

where $\alpha = c_1 A_{51} b_1$ and $\beta = c_1 A_{12} b_1$; but α is also the angle $a_{51} OL_1$, and β is $a_{12} OL_1$, while θ_1 is the angle between OL_1 and $a_{12} a_{51}$; hence this cotangent formula applied to the triangle $Oa_{12} a_{51}$ shows that OL_1 divides $a_{12} a_{51}$ in the given ratio $l_2^{(1)} : l_5^{(1)}$, the segment next a_{51} being proportional to $l_2^{(1)}$, and that next a_{12} to $l_5^{(1)}$. If we denote the length of $a_{51} a_{12}$ by p_1, and take the line $a_{51} a_{12}$ as axis of y, and a parallel to the bar $A_{51} A_{12}$ through the point a_{51} as axis of x, the equation of OL_1 is

$$y = \frac{l_2^{(1)}}{l_1} \cdot p_1.$$

If, in the same way, we take the line $a_{12} a_{23}$ as axis of y, and a parallel through a_{12} to the bar $A_{12} A_{23}$ as axis of x, the equation of OL_2 will be

$$y = \frac{l_3^{(2)}}{l_2} \cdot p_2,$$

where p_2 is the length of $a_{12} a_{23}$, and $l_3^{(2)}$ is the segment, $b_2 A_{23}$, of the bar $A_{12} A_{23}$ adjacent to A_{23} made by P_2. And similarly we have the equations of the other lines OL_3, OL_4, ... Transforming these into equations all referred to a common origin and axes, and expressing the condition that the co-ordinates of the point of intersection of any two must satisfy the equations of all the rest, we obtain $n-3$ new conditions of equilibrium of a deformable polygon formed of n bars, each acted on by an assigned force.

The analytical conditions are, however, more rapidly obtained by the Principle of Virtual Work.

Thus, supposing the system to be in equilibrium, choose a virtual displacement in which all the vertices except A_{12} and A_{23} remain at rest, i.e., let the bars $A_{51} A_{12}$ and $A_{34} A_{23}$ pivot round A_{51} and A_{34}, respectively. Remembering that the point of intersection, I, of $A_{51} A_{12}$ with $A_{34} A_{23}$ (produced) is the instantaneous centre for the bar $A_{12} A_{23}$, we find with no difficulty the equation

$$\frac{P_1}{l^{(1)}} l_5^{(1)} \sin \theta_1 \sin A_{23} + \frac{P_2}{l^{(2)}} \left[l_1^{(2)} \sin A_{12} \sin (\theta_2 - A_{23}) \right.$$
$$\left. - l_3^{(2)} \sin A_{23} \sin (\theta_2 + A_{12}) \right] + \frac{P_3}{l^{(3)}} l_4^{(3)} \sin \theta_3 \sin A_{12} = 0,$$

the angles of the polygon being denoted by the letters at the vertices.

Now $P_1 \sin \theta_1$ is the component of P_1 normal to the bar $A_{51} A_{12}$

measured outwards from the polygon, and $\frac{P_1}{l^{(1)}}$ would be the normal force per unit length if the normal component of P_1 were uniformly distributed along the bar, so that $P_1 \frac{l_5^{(1)}}{l^{(1)}}$ would be the total normal force over the segment $b_1 A_{51}$. Denote this by $N_5^{(1)}$. In like case, $P_3 \frac{l_4^{(3)}}{l^{(3)}}$ would be the total amount of normal force over the segment $b_3 A_{34}$, measured outwards from the polygon. Denote it by $N_4^{(3)}$. And, moreover, if ϕ_2 denotes the angle $I b_2 A_{23}$, while $\nu^{(2)}$ denotes the component of P_2 perpendicular to $I_2 b_2$ measured towards the same side of this line as that from which θ_2 is measured (i.e., the lower side of $I_2 b_2$ in the Figure), this equation becomes

$$\frac{N_5^{(1)}}{\sin A_{12}} - \frac{\nu^{(2)}}{\sin \phi_2} + \frac{N_4^{(3)}}{\sin A_{23}} = 0. \qquad (a)$$

The same type of condition is obtained for the three bars $A_{12} A_{23}$, $A_{23} A_{34}$, $A_{34} A_{45}$, and for each of the $n-2$ systems of three similarly taken in succession.

Special cases of deformable polygons of bars are treated independently in the examples following, and the student may verify the general results typified by (a) in these particular cases.

145.] Polygonal Framework. We shall now consider a *framework* of bars connected with each other by smooth pins at their extremities. The framework, moreover, is supposed to contain no superfluous bars, i.e., it contains just so many as are necessary to render its figure invariable. The principle of calculating the reactions of the bars in such a framework in general will be sufficiently understood from the discussion of the simple framework represented in Fig. 181. The graphic method here used is that which is usually employed in the calculation of the reactions in bridges; but in such structures the bars are so numerous that the figures of graphic statics which apply to them are extremely complicated, without, however, involving any more of principle than is involved in a simple framework consisting of only a small number of bars.

Fig. 181 represents a framework consisting of seven bars kept in equilibrium by forces applied at the vertices. These forces must of course satisfy the two graphic conditions of equilibrium, i.e., their force and funicular polygons must both be closed. Any three of them—suppose P_3, P_4, P_5—may be arbitrarily assumed both in magnitudes and in directions; and, in addition, we may assume the line of action of P_2. Then P_1 and P_2 are completely determinate, because we can construct the resultant of P_3, P_4, P_5 (Art. 94); produce the line of action of P_2 to meet this resultant; join their point of intersection to the vertex 1, and this will be the line of action of P_1. Hence P_1 and P_2 are both known.

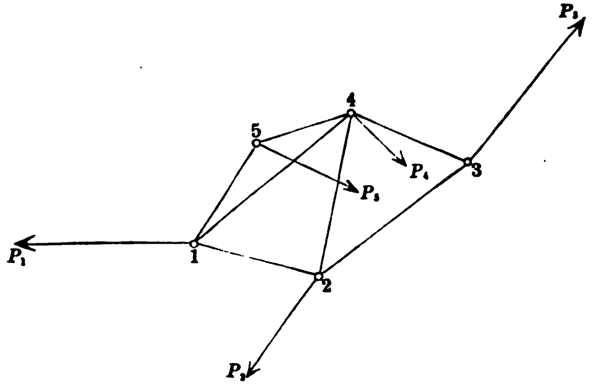

Fig. 181.

Having thus completely determined the external forces and drawn their force polygon, $a_{15}a_{12}a_{23}a_{34}a_{45}a_{15}$ (Fig. 182), we proceed to represent the equilibrium of each vertex separately.

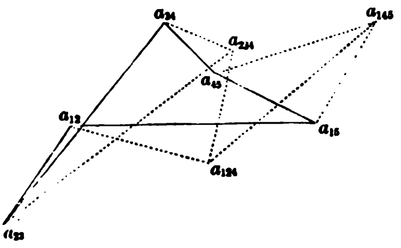

Fig. 182.

Each vertex is in equilibrium under the action of the external force at it and the reactions in the bars which meet in this vertex. Hence, it is obvious that the external force is completely equivalent to these reactions reversed, and therefore that it might be replaced by them in the force polygon of the external forces. If then we choose to replace any force—say P_4, which is represented by $a_{34}a_{45}$—by the reversed reactions of the bars which meet in the vertex, 4, it is clear that the lines representing P_4

and these reactions must, in Fig. 182, form a closed polygon. Denote this polygon by (P_4, ρ_4). Similarly the lines in Fig. 182, which represent P_5 and the reactions in the bars which meet in vertex 5, must also form a closed polygon, (P_5, ρ_5), say. And since one of these reactions—viz., that in the bar 45 which joins the vertices of P_4 and P_5—belongs also to the previous polygon, (P_4, ρ_4), these two polygons must have one side in common. But in the force polygon of the external forces, the forces P_4 and P_5 *have been drawn consecutively*, i.e., they have a point, a_{45}, in common; hence the side common to the two polygons must pass through this point; or, in other words, through the point of intersection of any two consecutive forces in diagram 182 must be drawn a parallel to the bar which connects their vertices in Fig. 181; and, moreover, no other line can pass through this point, because in drawing the polygon (P_4, ρ_4) only two forces are represented at each vertex, so that in this polygon we have P_4 and the reaction in bar 45 at the point a_{45}; and similarly only two forces are represented at this point in the polygon (P_5, ρ_5), viz., the force P_5 and the reaction just mentioned.

Hence through each vertex, $a_{12}, a_{23}, \ldots$, of the force polygon of the external forces pass *three* and only three lines.

But, in addition to the vertices $a_{12}, a_{23}, \ldots$ in Fig. 182 which correspond to two forces in Fig. 181 and the bar joining their vertices, there will be other vertices, $a_{124}, a_{234}, a_{145}$, through each of which pass also *three*, and only three, lines, viz., lines parallel to the bars forming the various triangles into which the framework (Fig. 181) is divided. That is to say, through a_{145} will pass three lines parallel to the sides of the triangle 145 of Fig. 181. It is easy to see that the line in Fig. 182, which answers to the bar 45 in Fig. 181, must pass through the point of intersection, a_{145}, of the lines answering to the bars 15 and 54 (Fig. 181). For we may replace the force P_5 by its two components along 51 and 54; and, imagining these components to act at the vertices 1 and 4, respectively, suppress altogether the bars 51 and 54. This would give us *two* external forces at the vertex 1, and also *two* external forces at the vertex 4; but the two external forces must at each vertex be replaced by *one*. If this is done, the external force at the vertex 1 would, in Fig. 182, be represented in magnitude and sense by the line $a_{145}\, a_{12}$, and the external force at vertex 4 would be represented by the line

$a_{34} a_{145}$; these forces would then be *consecutive* in the force polygon of external forces—which would be then $a_{145} a_{13} a_{23} a_{34} a_{145}$—and 14 (Fig. 181) would be the bar joining their vertices in the framework. But we have just seen that the line answering to the bar 14 should be drawn through the point, a_{145}, of intersection of the two (consecutive) external forces acting at vertices 4 and 1.

Hence, then—*Through each vertex in Fig.* 182 *pass three, and only three, lines, and these lines answer either to two consecutive forces and the bar joining their vertices, or to three bars forming a triangle in the framework.*

This removes any ambiguity which may arise in the construction of the diagram representing the reactions in the framework.

Thus, considering the equilibrium of the simplest vertices, viz., 5 and 3, in Fig. 181, we determine the reactions, $a_{34} a_{234}$, and $a_{45} a_{145}$, in the two bars 43 and 45 which belong to the vertex 4. Then for the equilibrium of this vertex we draw $a_{134} a_{34} a_{45} a_{145}$, reaching the point a_{145}, at which it is doubtful for a moment whether we are to draw a parallel to the force in the bar 42 or a parallel to the force in the bar 41—which are the two remaining forces acting on *B*. The consideration that the three lines to be drawn through a_{145} answer to three bars forming a triangle, and that since one of these lines, $a_{45} a_{145}$, already answers to bar 45 which belongs to the triangle 451, decides the question in favour of the bar 41. Thus the doubt is removed.

The force polygon of the external forces may be drawn in several different ways. Indeed, if our only object is to find the resultant of any number of forces acting on a rigid body, the sides of the force polygon may be drawn parallel to the forces in any order whatever. But in the calculation of the reactions in the bars of a framework we must observe the rule that in drawing the force polygon of the external forces *no two forces are to be drawn consecutively unless their vertices are connected by a bar.* In other words, two consecutive forces in the force polygon correspond to two forces at the vertices connected by a bar; but the converse does not hold—i.e., it is not true that the forces which act at every two connected vertices of the framework are consecutive in the force polygon.

For example, vertices 1 and 4 are connected, but their forces are not consecutive in Fig. 182.

The force polygon might have been constructed by drawing, in succession, lines to represent P_1, P_5, P_4, P_2, P_3. If the external forces are applied at given points in the bars, and not at the joints, they may all be replaced by components at the joints, as explained in Art. 142, and the calculation of the reactions which would thence result proceeds graphically as explained in this Article. The true reactions exerted at the extremities of each bar in the actual case—reactions which are not directed along the bars—can then be found as explained in Art. 142.

The student who is desirous of studying the force diagrams of systems of frameworks, such as those which belong to Engineering and Architecture, is recommended to study, in the first instance, Levy's *Statique Graphique*. An excellent work, treating very fully of the theory of reciprocal figures and their statical applications, is Favaro's *Lezioni di Statica Grafica* (Padova, 1877). For great elaboration of the subject Culmann's *Die Graphische Statik* may be consulted.

146.] Method of Separation of the Bars. Another method, which is often convenient in practice, consists in representing the bars as disjointed from each other, and replacing the reactions by rectangular components, parallel to chosen axes, at their extremities. Two examples will suffice. Four equal uniform bars, *AB*, *BC*, *CD*, and *DE* (Fig. 183), are connected by smooth pins at *B*, *C*, *D*, and the extremities, *AE*, are fixed in a horizontal line by smooth joints; it is required to find the position of equilibrium.

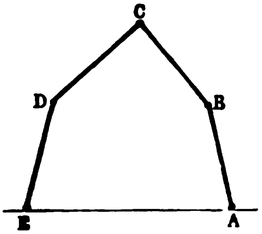

Fig. 183.

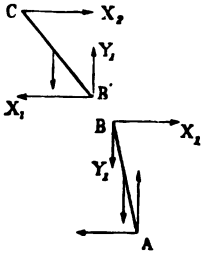

Fig. 184.

Let α be the common inclination of *AB* and *ED* to the horizon, and β that of *CB* and *CD*.

Let Fig. 184 represent the bars *AB* and *BC* separated; X_2 the reaction at *C*, which is evidently horizontal; X_1 and Y_1 the components of the reaction at *B*. These components act on *AB* in directions opposite to those in which they act on BC. Finally, let W be the weight of each bar.

Resolving vertically for the equilibrium of BC,

$$Y_1 = W. \tag{1}$$

Taking moments about C for the equilibrium of BC,

$$2X_1 \sin\beta + W\cos\beta = 2Y_1 \cos\beta,$$

or

$$X_1 = \tfrac{1}{2}W\cot\beta. \tag{2}$$

Taking moments about A for the equilibrium of AB,

$$(W + 2Y_1)\cos\alpha = 2X_1 \sin\alpha,$$

or, substituting the values of X_1 and Y_1 from (2) and (1),

$$\tan\alpha = 3\tan\beta. \tag{3}$$

With this equation must be combined the geometrical equation which expresses that AE is equal to the sum of the horizontal projections of the bars. If the length of each bar is a, and the distance $AE = c$, we have

$$c = 2a(\cos\alpha + \cos\beta). \tag{4}$$

Equations (3) and (4) determine α and β, and therefore the position of equilibrium.

Graphically, α and β can be found from the intersection of a right line and a magnetic curve.

When the bars are numerous, the use of a large number of symbols, X_1, Y_1, X_2, ... for the components of reactions necessitates the writing down of a large number of equations of equilibrium, many of them consisting of only two terms. Hence a saving of symbols and equations is easily effected by making use of the simple results of resolution of forces as we proceed from joint to joint and bar to bar, so that finally we have to use equations of moments almost entirely. This process is illustrated in the solution of the following problem. $ABCD$ (Fig. 185) is a hexagon formed of equal bars, each of weight W; it is suspended in a vertical plane from the pin A which is fixed, and the hexagon is kept regular by cords connecting A with C, D, and E. Find the tensions of these cords and the reactions at the various joints.

We shall assume that the joints are formed by pins distinct from the bars themselves, though this is not absolutely necessary. Fig. 186 represents the framework completely disjointed, and exhibits the separate equilibrium of each pin and bar.

Considering the equilibrium of the framework, it is kept at rest by two forces, viz., its weight, $6W$, and the reaction of the wall in which the pin A is fixed; hence this reaction $= 6W$. Consider the separate equilibrium of the pin A. It is at rest under the influence of this vertical upward reaction, the tensions, S, in the cords AC, AE, the tension T in AD, and the reactions of the bars AB, AF, of which nothing is yet known.

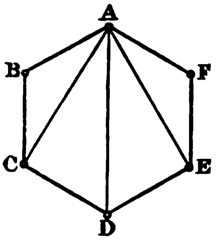

Fig. 185.

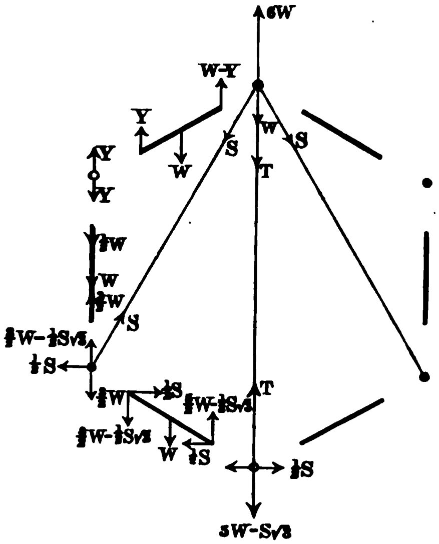
6W
W-Y
Y
W
S
W
S
T
Y
Y
½W
W
½W
S
$\frac{3}{2}W-\frac{1}{2}S\sqrt{3}$
½S
½W
½S
$\frac{3}{2}W-\frac{1}{2}S\sqrt{3}$
T
$\frac{3}{2}W-\frac{1}{2}S\sqrt{3}$
W
½S
½S
$3W-S\sqrt{3}$

Fig. 186.

Let X and Y be the horizontal and vertical components of the reaction of the pin B on the bar AB. These reversed are the forces exerted on the pin B by the bar AB; and therefore X and Y are the components of reaction of the bar BC on the pin B (from consideration of separate equilibrium of the pin B). Hence the pin B exerts on the bar BC the components X, Y in the senses opposite to those in which these components act on AB. But from the equilibrium of the bar BC we see at once that $X = 0$, since BC is vertical and all the forces acting on it, except X, have zero moments about C. This now enables us to see that $Y = \frac{1}{2}W$ by considering the equilibrium of the bar AB: for if Y is the force acting on it at B, the force at A must be $W - Y$, and moments about either A or B give $Y = \frac{1}{2}W$.

Henceforth in the figure, then, we may discard the symbol Y and use $\frac{1}{2}W$ for it.

Consider the equilibrium of the bar BC. It is acted upon downwards by W, and by $\frac{1}{2}W$ at B; therefore the pin C is producing on it a vertical upward force $= \frac{3}{2}W$, and no horizontal force.

Consider the equilibrium of the pin C. It is acted upon downwards by $\frac{3}{2}W$ from the bar BC, by the tension S, and by the reaction of the bar CD, whose components must be $\frac{1}{2}S$ towards the left and $\frac{3}{2}W-\frac{1}{2}S\sqrt{3}$ vertically up; and these last components reversed are those exerted by the pin C on the bar CD, the other forces keeping which in equilibrium are W (its weight) and the reaction of the pin D, whose components must, therefore, be $\frac{1}{2}S$ towards the left and $\frac{5}{2}W-\frac{1}{2}S\sqrt{3}$ vertically up. These last reversed are the components of the force exerted on the pin D by the bar CD.

Finally, considering the equilibrium of the pin D, we see that it is acted upon by $\frac{1}{2}S$ towards the right and $\frac{5}{2}W-\frac{1}{2}S\sqrt{3}$ downwards from the bar CD, and (by symmetry) by $\frac{1}{2}S$ towards the left and $\frac{5}{2}W-\frac{1}{2}S\sqrt{3}$ downwards from the bar DE, as well as by the tension T. Resolving these forces vertically, we have

$$T = 5W-S\sqrt{3},$$

which is precisely the same as the result obtained by considering the separate equilibrium of the pin A, because we have shown that the bar AB exerts on it a vertical downward force $= W-Y$, i.e., $\frac{1}{2}W$, and no horizontal force. Similarly the bar AF exerts $\frac{1}{2}W$ on it vertically downwards.

The unknown force S is found from the equilibrium of the bar CD by moments about D, from which we have

$$S = W\sqrt{3},$$

and therefore $$T = 2W.$$

Thus, then, the reaction at C on the bar CD is wholly horizontal and $= \frac{1}{2}W\sqrt{3}$, the reaction at its end D passing through the intersection of the lines of action of this force and the weight of the bar, and being equal to $\frac{1}{2}W\sqrt{7}$.

Examples.

1. A triangular system of bars, AB, BC, and CA, freely jointed at their extremities, is kept in equilibrium by three forces acting on the joints; determine the reactions at the ends of each bar.

Since the forces are applied directly to the joints, the reactions will act along the bars. Let P, Q, R, denote the forces applied at A, B, C, respectively; let the reactions in the sides BC, CA, AB be denoted by T_1, T_2, T_3; and let the applied forces meet in a point O.

Then for the equilibrium of the joint C, we have

$$\frac{T_1}{T_2}=\frac{\sin ACO}{\sin BCO}=\frac{a.OA.\sin AOC}{b.OB.\sin BOC},$$

a, b, c being the sides of the triangle.

But $P:Q:R=\sin BOC:\sin COA:\sin AOB$. Therefore

$$\frac{T_1}{T_2}=\frac{a.OA.Q}{b.OB.P}, \text{ or}$$

$$T_1:T_2:T_3=\frac{a.OA}{P}:\frac{b.OB}{Q}:\frac{c.OC}{R}.$$

If O is the centroid of the triangle, we know (p. 155) that

$$P:Q:R=OA:OB:OC;$$

therefore $$T_1:T_2:T_3=a:b:c,$$

or the reactions are proportional to the sides.

If O is the orthocentre (or intersection of perpendiculars),

$$P:Q:R=a:b:c;$$

therefore $$T_1:T_2:T_3=OA:OB:OC.$$

2. A number of bars are jointed together at their extremities and form a polygon; each bar is acted upon perpendicularly by a force proportional to its length, and all these forces emanate from a fixed point. Find the magnitudes and directions of the reactions at the joints.

[This problem and the following elegant method of solution are due to Professor Wolstenholme.]

Let AB and BC (Fig. 187) be any two adjacent bars of the polygon, and let P be the point from which emanate the forces, Pp, Pq, ..., acting on the bars. Then the reactions at the joints A and B, acting on AB, must meet in a point, p, on the line of action of the force Pp. Draw AQ and BQ perpendicular to the reactions in the directions Ap and Bp. Now since the sides of the triangle AQB are perpendicular to three forces which are in equilibrium, and since the side AB is proportional to the force to which it is perpendicular, the sides AQ and BQ are proportional to the forces to which they are perpendicular, that is, to the reactions at A and B, respectively.

Let q be the point in which Bp intersects Pq. Then the forces acting on the bar BC must act in the directions qB, Pq, and qC. Draw CQ.

In the triangle BQC the sides BQ and BC are perpendicular and proportional to two of three forces in equilibrium; therefore CQ is perpendicular and proportional to the third, that is, to the reaction at C. In the same way it can be shown that the reaction at any joint is perpendicular and proportional to the line joining the joint to Q. This point Q is, therefore, a *reaction centre* for the system. It may be shown that *the polygon of bars must be inscribable in a circle*. For, since the angles at A and B are right, the quadrilateral $ApBQ$ is inscribable in a circle whose diameter is pQ. If at the middle point of AB a perpendicular be drawn to AB, it will pass through the centre of the circle, and will, therefore, bisect Qp. But this perpendicular is parallel to Pp; therefore it bisects PQ in O. Also, since the reactions at A and B are proportional to QA and QB, the same point Q must be determined by considering BC and the next bar, as was determined from the bars AB and BC; consequently the point O must be the same; and since it is evident that $OB = OC, \ldots$, O must be equally distant from all the vertices of the polygon, that is, the polygon must be inscribable in a circle.

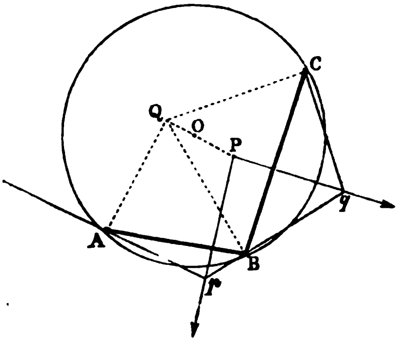

Fig. 187.

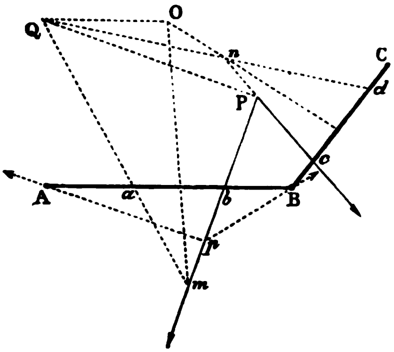

Fig. 188.

The reaction centre is therefore constructed by joining P to the centre of the circumscribing circle, and producing PO to Q, so that $PO = OQ$.

3. The preceding construction can be extended to the case in which the forces acting on the polygon are equally inclined, but not perpendicular, to the sides.

Let AB, $BC, \ldots$ be sides of the polygon, and let forces proportional to the sides act in the lines Pb, $Pc, \ldots$ so that $\angle PbB = \angle PcC = \ldots$. It is required to prove that for equilibrium the polygon must be inscribable in a circle, and to find the reaction centre.

The reactions at A and B must meet in a point on the force in Pb. If, then, we draw at A and B lines, QA and QB, making with the directions of the reactions angles equal to $\angle PbB$, we shall have a triangle, QAB, the sides of which are each equally inclined to the corresponding force; and, since AB is proportional to the force in Pb, it follows that QA and QB are proportional to the reactions at A and B. It is easy to prove that if through A and B any two lines, Ap and Bp, be drawn, meeting in a point on the right line Pb; and at A and B lines, AQ and BQ, be drawn making with Ap and Bp, respectively, angles equal to PbB, the locus of Q is a right line, ma, making $Aa = Bb$, and $\angle maB = \angle mbA$. Drawing the line Qd, in like manner, by making $Cd = Bc$ and $\angle QdB = PcC$, we obtain the point Q, which is the reaction centre.

Now, since $\angle PcC = \angle PbB$, it follows that $\angle bPc$ is the supplement of $\angle B$; and since $\angle QaA = \angle QdB$, it also follows that $\angle aQn = \pi - B$. Hence the quadrilateral $mPnQ$ is inscribable in a circle, and this circle must pass through O, the point of intersection of the perpendiculars to AB and BC drawn at their middle points, since $\angle mOn$ is also the supplement of B. Hence also

$$\angle QPO = \angle QnO = \frac{\pi}{2} - ncC, \quad \text{and} \quad QO = OP.$$

Again, the reactions at A and B being proportional to QA and QB, the same point Q must be determined when BC and the next bar are considered. Hence the point O is the same. But

$$OA = OB = OC = \ldots;$$

therefore the polygon is inscribable in a circle.

The point P being given, if the angle which the forces through it make with the corresponding bars varies, the locus of the reaction centre, Q, is a circle concentric with that round the polygon, its radius being OP. To construct the reaction centre, then, we describe a circle round O as centre, having radius OP, and draw PQ making the $\angle OPQ =$ the complement of the angle which the forces make with the bars.

4. A system of heavy bars, freely articulated, is suspended from two fixed points, P and Q (Fig. 189); determine the magnitudes and directions of the reactions at the joints.

Let the bars be denoted by the numbers 1, 2, 3, ..., and let their weights be W_1, W_2, W_3, ... Then transfer $\frac{1}{2}W_1$ and $\frac{1}{2}W_2$ to the joint connecting 1 and 2, which we shall denote by (1, 2). Transfer $\frac{1}{2}W_2$ and $\frac{1}{2}W_3$ to the joint (2, 3); $\frac{1}{2}W_3$ and $\frac{1}{2}W_4$ to (3, 4), &c. Thus all the forces act at the joints. Let T_1, T_2, T_3,... be the tensions acting along the bars 1, 2, 3,... on the joints, and let S_{12}, S_{23}, S_{34},... be the total reactions at the joints (1, 2), (2, 3), (3, 4),.... For

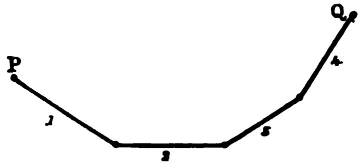

Fig. 189.

simplicity suppose the bar 2 to be horizontal. Now, construct a force-diagram (Fig. 190), by drawing a vertical line, AD, and measuring off

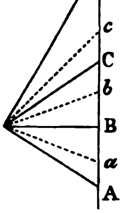

Fig. 190.

$$AB = \frac{W_1 + W_2}{2},\quad BC = \frac{W_2 + W_3}{2},\quad CD = \frac{W_3 + W_4}{2}\ldots.$$

Also take BO parallel to the bar 2 and equal to the tension T_2, which is the constant horizontal component of each of the tensions. The lines OA, OC, OD,... will then be parallel to the bars 1, 3, 4,... and equal to the tensions in them. Hence, if α be the inclination of 3 to the horizon,

$$\tan DOB = \frac{W_2 + 2W_3 + W_4}{W_2 + W_3} \cdot \tan\alpha,$$

and in the same way the inclinations of the other bars may be expressed in terms of the inclination α.

Again (Art. 142), the reaction S_{12} is the resultant of T_1 and $\frac{W_1}{2}$. Hence, taking $Aa = \frac{W_1}{2}$, Oa will be equal and parallel to S_{12}. Similarly, taking $Bb = \frac{W_2}{2}$, and $Cc = \frac{W_3}{2}$, the lines Ob and Oc will be equal and parallel to the reactions S_{23} and S_{34}. The tangent of the angle made by S_{23} with the horizon $= \frac{bB}{BO} = \frac{W_2}{W_2 + W_3} \cdot \tan\alpha$. Similarly for the directions of the other reactions.

If the weights of the bars are all equal, the tangents of the inclinations of the successive bars are $\tan\alpha$, $2\tan\alpha$, $3\tan\alpha$,... and the tangents of the inclinations of the reactions are $\frac{1}{2}\tan\alpha$, $\frac{3}{2}\tan\alpha$, $\frac{5}{2}\tan\alpha$,....

5. A rectangular frame, $ABCD$, consists of four bars, whose weights are negligible, freely jointed at their extremities, the bar AD being completely fixed in a vertical position, AB being the upper and CD the lower bar. If a load W is placed at a given point, P, on AB, while the rectangular figure is preserved by a cord connecting the joints A and C, find the tension of this cord and the stresses in the bars. If the load W is transferred to the point on CD vertically under P, prove that the tension is unaltered, and find the stresses. (*Scholarship Examination, Cambridge*, 1886.)

Ans. If $AP = x$, $AB = a$, $BC = b$, $AC = c$, we have the tension $= W\frac{cx}{ab}$. In the first case the stress in BC consists of pressure equal to $W\frac{x}{a}$, and in CD of pressure equal to $W\frac{x}{b}$. In the second case, if the bars are connected by pins distinct from them, there is no stress in AB or BC, while the action of the pin C on the bar CD is the tension of the cord.

6. Six equal uniform bars, freely articulated at their extremities,

form a hexagon $ABCDEF$ (Fig. 191). The bar ED is fixed in a horizontal position, and its middle point is connected by a string with the middle point of the lowest bar, AB, in such a manner that the bars hang in the form of a regular hexagon. Find, by a force-diagram, the tension of the string and the magnitudes and directions of the reactions at B and C.

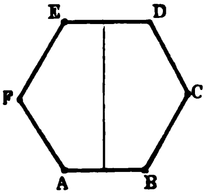

Fig. 191.

Ans. If W is the weight of each bar, the tension of the string $= 3W$; the reaction at C is horizontal, and $= \frac{W}{2\sqrt{3}}$; the reaction at $B = W\sqrt{\frac{13}{12}}$, and makes with the horizon an angle whose tangent $= 2\sqrt{3}$.

7. Prove that the reaction centre for the bar BC is the intersection of a perpendicular to BC at C with the line joining the middle points of AB and BC.

8. Three bars, freely articulated, form a triangle ABC, the centre of whose inscribed circle is O. Each bar is acted on by a force passing through O, proportional to the sine of half the angle subtended by the bar at O, and bisecting this angle. Prove that the reaction at A makes with OA an angle whose tangent is

$$\frac{\sin\frac{A}{2}}{\cos\frac{B}{2} - \cos\frac{C}{2}}.$$

(This is a direct example of the Theorem of Art. 142.)

9. AB (Fig. 192) is a rigid bar whose weight is neglected fixed at one extremity, A, by a smooth joint; CD is another such bar fixed at C by a smooth joint, which is vertically below A, and jointed to AB at D. From B a mass of weight, P, is suspended; find the magnitudes and directions of the reactions at the joints.

Ans. The reactions at C and D are along CD, and each $= P\cdot\frac{AB\,.\,CD}{AC\,.\,AD}$; the reaction at A is in AO, O being the intersection of CD produced with the vertical through B, and

$$= P\cdot\frac{\sqrt{AB^2\,.\,AD^2 + AC^2\,.\,BD^2}}{AC\,.\,AD}.$$

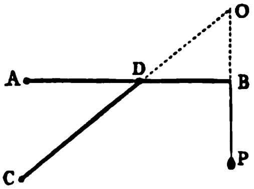

Fig. 192.

10. In example 6, if the bars BC and CD, AF and FE, are replaced by any bars all equally inclined to the horizon, show that the reactions at C and F will still be horizontal.

[One simple proof of this is obtained by taking moments about B for the equilibrium of BC, and about D for the equilibrium of CD. It follows then that the perpendiculars from B and D on the line of action of the reaction at C are equal.]

11. Two uniform heavy bars are freely jointed at a common extremity, and are fixed at their other extremities to two smooth joints in a vertical line; find the reactions at the joints.

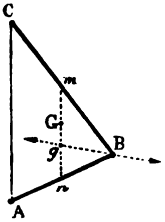

Fig. 193.

Ans. Let G (Fig. 193) be the centre of gravity of the bars, m and n their middle points. It follows, by taking moments about A and C for the equilibrium of the bars separately, that the segments of AC made by the line of the reaction at B are proportional to the weights of the bars. Hence, taking $ng = mG$, the reaction acts in the line gB. The reactions at A and C act, therefore, in Ag and Cg. If W is the weight of AB, the reaction at $B = \frac{1}{2} W \frac{gB}{gn}$, and the reaction at $A = \frac{1}{2} W \frac{gA}{gn}$. Hence the reactions at A, B, and C are proportional to gA, gB, and gC.

12. Five equal bars, each of length a, are hinged together and placed on a smooth horizontal table, one of the angular points being joined to the two opposite vertices by two equal strings of length $2c$; horizontal forces, each equal to P, act at the middle points of all the rods in an outward direction perpendicular to them; find the tensions of the strings, and show that the stress at each of the joints where there is no string is parallel to the nearest string, and equal to

$$P \frac{a}{2\sqrt{a^2-c^2}}.$$

(*Tripos*, 1887.)

Ans. Let A, B, C, D, E, be the vertices, A being connected with C and D; let $\angle ACD = \alpha$, $\angle ACB = \beta$; if T is the tension of each string, we have

$$T = -\frac{1}{4} P (\operatorname{cosec} \alpha + 2 \operatorname{cosec} \beta + 4 \sin \beta),$$

which shows that *strings* will not suffice, so that A must be connected with C and D by *bars*.

The most simple way of seeing that the stress at B is parallel to AC is to consider the separate equilibrium of the bar AB and take moments about A. The moment of the stress about A is equal to $\frac{1}{2} Pa$. Again, consider the separate equilibrium of the bar BC, and take moments about C. The moment of the stress at B about C is again $\frac{1}{2} Pa$, so that the perpendiculars on its line of action from A and C are equal, $\therefore$ it acts in a line parallel to AC; whence its magnitude follows at once.

13. Six bars of equal lengths and weights are freely jointed together at their extremities and form a hexagon which is hung vertically from one corner; the two middle rods are connected by a horizontal bar whose weight is negligible. Show that if this bar is so attached that

the two middle bars rest vertically, the ratio of the segments into which it divides them is 5 : 1, no matter what the figure of the hexagon is. If the horizontal bar has weight equal to that of each bar, the ratio is 6 : 1. Find the longitudinal stress in the connecting bar.

Ans. If 2θ is the angle at the fixed corner, the compressive stress is, in the first case, $3W\tan\theta$; and, in the second, $\frac{7}{2}W\tan\theta$.

14. The regular hexagon of bars in example 6 rests in a vertical plane, the bar AB being fixed in a horizontal position, and the joints F and C are connected by a string; find the tension of the string, and the reactions acting on the bar FE at its extremities.

Ans. The tension $= W\sqrt{3}$ (W being the weight of each bar); the reaction at $E = \dfrac{W}{2}\sqrt{\dfrac{7}{3}}$, and it makes with FE $\sin^{-1}\frac{1}{2}\sqrt{\dfrac{3}{7}}$; the reaction at $F = \dfrac{W}{2}\sqrt{\dfrac{31}{3}}$, and it makes with FE $\sin^{-1}\frac{1}{2}\sqrt{\dfrac{3}{31}}$.

15. Four equal uniform heavy bars, freely jointed together at their extremities, form a square, $ABCD$; the joint A is fixed, while the diagonally opposite joints B and D are connected by a string, and the whole system rests in a vertical plane, the string being horizontal; find the tension of the string and the magnitudes and directions of the reactions on the bars at A, B, and C.

Ans. The tension $= 2W$; the reaction at C is horizontal and $= \frac{1}{2}W$; the reaction on the bar BC at B makes with the vertical $\tan^{-1}\frac{1}{2}$, and $= W\dfrac{\sqrt{5}}{2}$; the reaction on AB at B makes with the vertical $\tan^{-1}\frac{3}{2}$, and $= W\dfrac{\sqrt{13}}{2}$; and the reaction on AB at A intersects the line BD at a distance $\frac{1}{8}BD$ from B, and is equal to $\frac{5}{2}W$.

16. Two uniform heavy bars, AB and BC, connected by a smooth joint at B, rest each on a smooth vertical prop, the props being of the same height; find the position of equilibrium, ABC being horizontal.

Ans. If W and $2a$ are the weight and length of AB, W' and $2b$ the weight and length of BC, c the distance between the props; then x, the distance of the middle point of AB from the corresponding prop, is given by the equation

$$(W+W')x^2+[(W+W')(c-a)-W'(a+b)]x-W'a(c-a-b)=0.$$

17. Two bars, AC, CB, connected together by a smooth horizontal axis at C, rest with their extremities A and B on a horizontal plane, a tight cord, AD, being attached to one bar at A and to the other at any point, D, in its length; determine the line of resistance at C, and show that if the point D is made to vary along CB, the point of intersection of the line DA and the line of resistance will describe a given vertical line.

(Let W be the weight of AC, and let the line of action of W meet AB in m; let W' be the weight of BC and let its line of action meet

AB in m'. Let AD meet the vertical at B in Q; in DA produced take O such that $\frac{QO}{OA} = \frac{W'.Bm'}{W.Am}$; then OC is the line of resistance. As D varies, this ratio is constant. The resistance is least when D is at B.)

18. ABC is an isosceles triangular framework of heavy bars jointed together at the vertices, the equal sides are AC and BC; the extremities A and B rest on two smooth vertical pillars of equal height, the plane of the triangle being vertical; a mass P is suspended from the joint C; find the reactions at the joints.

Ans. Let $W =$ weight of $AC =$ weight of BC; $a = \angle CAB$; then the reaction on AC at C makes with AC an angle θ such that $\tan\theta = \frac{W \sin a \cos a}{P + W\cos^2 a}$, the horizontal and vertical components of this reaction being, respectively, $\frac{1}{2}(P+W)\cot a$ and $\frac{1}{2}P$. The reaction between the bars AC and AB has for horizontal and vertical components, respectively, $\frac{1}{2}(P+W)\cot a$ and $W+\frac{1}{2}P$.

19. Three uniform bars, AB, BC, CD, freely jointed at B and C, are attached by smooth hinges to two points A and D in the same horizontal line, the lengths of AB and CD being equal; a fourth uniform bar, EF, rests horizontally against AB at E and against CD at F; find the reactions at the joints and hinges.

Ans. Let $a = \angle DAB$; $AB = CD = 2a$; $BC = 2b$; $EF = 2c$; $P =$ weight of $AB =$ weight of CD; $Q =$ weight of BC; $W =$ weight of EF. Then the horizontal component of reaction at A

$$= \frac{W}{\sin 2a}\left(\frac{c-b}{2a\cos a} - \cos^2 a\right) - \frac{1}{2}(P+Q)\cot a;$$

vertical component of reaction at $A = P+\frac{1}{2}(Q+W)$; horizontal component at $B = \frac{1}{2}W\tan a -$(previous horizontal component); vertical component at $B = \frac{1}{2}Q$.

20. A plane polygon is formed of bars jointed together at their extremities, and forces are applied at the joints; show that there is one and only one set of equilibrating forces meeting in a point which will cause given stresses in three selected bars. (*Tripos*, 1888.)

(Since the polygon is a funicular of the forces, the point of intersection of any two sides lies on the resultant of the intermediate forces. If the stresses in the two sides are given, the resultant of the intermediate forces is given in magnitude and line of action. Similarly for one of these sides and any other; and all the forces are given concurrent.)

21. $ABCD$ is a jointed quadrilateral at rest under two pairs of equal and opposite forces, viz., P, P applied to the sides AB and CD, and Q, Q applied to the sides AD and BC. If the reactions at the opposite joints are parallel to each other, prove that they are proportional to the sides of the parallelogram along which they act, and that the applied forces, P, Q, are proportional to its diagonals.

22. A quadrilateral, *ABCD*, of bars freely jointed together at their extremities is to be held in equilibrium in a given figure by four forces, P_1, P_2, P_3, P_4, each of which is to be applied to a side of the quadrilateral. If the forces P_1 and P_2, applied to the sides *AB* and *BC*, respectively, be completely given, show that the reactions at all the joints except *D* are completely given, and find to what extent the remaining two forces, P_3, P_4, are arbitrary.

Ans. The remaining forces may have any magnitudes whatever, provided that their arithmetical sum is greater than the magnitude of the resultant of P_1 and P_2; but, once the magnitudes of P_2 and P_3 are assumed, their lines of action are determined, and so, of course, is the reaction at the joint *D*. (Consider Fig. 180.)

CHAPTER X.

EQUILIBRIUM OF ROUGH BODIES UNDER THE INFLUENCE OF FORCES IN ONE PLANE.

147.] **Criterion of the existence of Friction.** We have already learned to regard Friction as a *passive resistance*; and every passive resistance comes into existence for the purpose of preventing some motion. Thus, the normal reaction of a surface on a body in contact with it comes into existence for the purpose of preventing the body from penetrating the surface at the point of contact; and if the circumstances of the case were so arranged that there was no tendency to this penetration, the magnitude of the force (normal resistance) required to prevent this motion would be zero.

Friction comes into existence for the purpose of preventing a certain motion—motion in the tangent plane—of a body resting against a rough surface. If the circumstances in any case of two rough bodies in contact are such that there is no tendency to slip at their point of contact, the force required to prevent this motion (friction) will not come into existence.

Generally, in the case of all passive resistances, *if there is no tendency to the displacement which a passive resistance is required to prevent, this force will not come into play.*

Hence in many cases of contact between rough bodies the conditions and circumstances are exactly the same as if the bodies were smooth; and to find whether in the contact of two bodies friction acts or not—*imagine that the bodies were smooth at their point of contact, and if no displacement would result from this supposition, friction does not come into play at that point.*

In illustration of this consider the problem in Example 26, p. 174. How would the circumstances be altered if the peg Q were rough?

The peg being rough, let it be imagined to become smooth, and what motion occurs? Clearly none, supposing the board to be rigid. Hence as there is no tendency of the side *AB* to slip over the peg, there is no friction called into play, and the case is the same as if the peg were smooth. But if the board is not rigid, the forces acting can bend its fibres and elongate or contract them; and if we imagine the peg to become smooth, it is possible that (even a very slight) slipping might ensue at the peg, and as this slipping is prevented by the roughness, the force of friction really acts in the case, and the pressure on the hinge is modified by the assumption of smoothness at the peg.

However, even when the board is elastic, it is possible that no friction is called into play, as will be explained in Art. 154.

Rankine's hint that friction is of the nature of shearing stress has been already pointed out.

148.] **The Cone of Friction.** The essential characteristic of a smooth surface is that it is capable of resisting in a normal direction only. If two rough surfaces are in contact, their mutual reaction is not constrained to assume a direction normal to the surface of contact. Each surface is capable of offering resistance to the other in any direction which does not make with the normal to the surface of contact an angle exceeding a certain magnitude. Thus (Fig. 194), let two rough bodies, *A* and *B*, be in contact at any point, *P*, and let *PN* be the normal to the surface of contact.

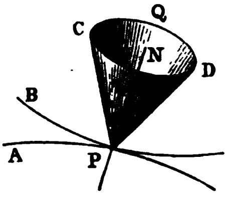

Fig. 194.

Let λ denote the greatest angle that the total resistance at *P* can make with *PN*, or, in other words, the greatest obliquity of the mutual reaction; then, describing round *PN* a right cone, *CQD*, whose semivertical angle, *NPD*, is equal to λ, this cone is called the *cone of friction*, and the total resistance at *P* can act in any direction whatever included within this cone. This angle λ is what we have called in Chap. III the *angle of friction*, and its tangent is the *coefficient of friction* for the two surfaces considered. For, if *N* denote the normal pressure between them at *P*, and *F* the force of friction (which acts in the common tangent plane), it is clear that when the resultant of *N* and *F* acts along any generator, *PD*, of the cone, we have

$$\frac{F}{N} = \tan NPD = \tan \lambda,$$

so that tan λ is the greatest ratio of the force of friction to the normal pressure. This quantity we have called μ.

If a rigid rod, M (p. 52), of negligible weight, be pressed against a rough surface at O, the greatest angle that the rod can make with the normal is the angle of friction. For, since the rod is acted on by only two forces, viz., the applied pressure and the total resistance at O, these must be equal and opposite, that is along the rod. Hence the greatest obliquity of the rod to the normal is λ.

If the resistance to slipping is not the same in different azimuths, i.e., if it is different in different planes through the normal, the value of λ will not be the same in all these planes, and the cone of friction will not be a right circular cone.

149.] **Axiomatic Law of Friction.** We have said that the total resistance of a rigid surface is a force which can assume any magnitude. This force will in any given case be exerted by the surface to such an extent as is necessary to preserve equilibrium, but to no greater extent. It is in its nature a *passive resistance*, i.e., one which can be exerted to any extent, but which will not be exerted beyond the bare requirements of the case. Within certain limits, also, it can assume any direction, and in any given case it will, if possible, assume such a direction as will preserve equilibrium. In fact, in virtue of its *passive* nature, we must regard the resistance of a rough surface as an opposition called into existence by the action of external forces; and it is evident that these forces will call into play only that amount of opposing force, exact both in magnitude and in direction, which will just counteract their own action.

The amount of assumption contained in this principle is enunciated in the following axiom :—

The total resistance which acts at any point of a rough surface will, if possible, assume such a magnitude and direction as will preserve equilibrium at that point.

This axiom is sometimes expressed thus :—*If passive resistances can give equilibrium, they will.*

150.] **Remarks on this Axiom.** Two important observations must be made on the principles contained in this axiom. Firstly, it is important to understand the circumstances which may render it impossible for the resistances of rough surfaces to preserve the equilibrium of a system in any given position.

Suppose that a body, acted on by given external forces, is in contact with a rough surface at a single point, *P*. Then, for equilibrium, it is necessary that the resultant of the given external forces should pass through *P*, and that the total resistance at *P* should be equal and opposite to this resultant. But if the direction of the resultant makes with the normal to the surface of contact at *P* an angle $> \lambda$, it is impossible that the total resistance could take the required direction, and equilibrium cannot subsist.

Again, take the case in which a heavy beam, *AB* (Fig. 195), rests against a rough horizontal and an equally rough vertical plane. Describe round the normals to the planes at *A* and *B* the cones of friction, and let the sections of these cones by the plane of the figure be *rAs* and *pBs*. Let *G* be the centre of gravity of the beam, and *GV* the vertical line through it.

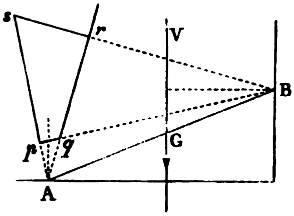

Fig. 195.

Then the beam, if in equilibrium, is so under the action of three forces, namely, the weight through *G* and the total resistances at *A* and *B*. These three forces must meet in a point, and if it be possible to find a point in which they can meet, the resistances will assume proper values. Now, in the figure it is impossible to find any point on *GV*, the line of action of the weight, the lines drawn from which to *A* and *B* could be directions of possible resistance at *both A* and *B*. For the portion of *GV* which is inside the cone of friction at *B* is outside the cone of friction at *A*, and *vice versâ*. Hence, for equilibrium, *there must be some portion of the line GV included in the space pqrs, common to both cones of friction.*

Unless this condition is satisfied, it is not possible for the total resistances to give equilibrium, whatever their magnitudes may be. A possible position of equilibrium is represented in Fig. 196. For, if from any point on the portion, *mn*, of *GV*, which is included in the space common to both cones of friction, lines be drawn to *A* and *B*, these lines are possible directions of total resistance at *A* and *B*; and in this case the actual magnitudes and directions of the resistances at *A* and *B* cannot be determined by what is called Rational Statics.

If it be proposed to find the position of *limiting* equilibrium, that is, the position in which the beam is bordering on motion,

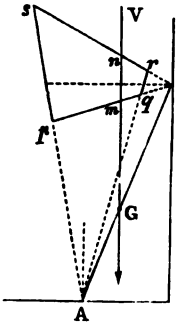

Fig. 196.

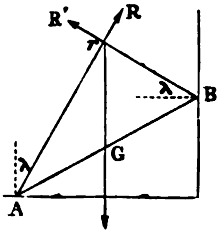

Fig. 197.

we must make the vertical through G pass through r, as in Fig. 197.

In this case there is only one point on GV which is inside both cones of friction, viz., the point r. Hence the total resistances act in rA and rB, and each makes the limiting angle (λ) with the corresponding normal. Moreover, both resistances are now determinate. If θ be the angle made by the beam with the horizon, we have, from the triangle ArB,

$$2 \cot rGB = \cot ArG - \cot BrG,$$

or

$$2 \tan \theta = \cot \lambda - \tan \lambda,$$

$$= \frac{1-\mu^2}{\mu};$$

which defines the position of limiting equilibrium.

It may, therefore, in certain cases be impossible for the total resistance at one or more points to preserve equilibrium; and this impossibility is always due to something in the arrangement of the figure or the external forces which requires the direction of the resistance to make with the normal to the surface of contact an angle > the angle of friction.

Again, in the axiom is contained the following important proposition :—

If a body rests against a rough surface at a point, and if the equilibrium is about to be broken by some change in the acting forces, equilibrium at that point will, if possible, be broken by a rolling instead of a sliding motion.

For, in this case, the point of the body actually in contact with the surface would be kept at rest. This part of the axiom is sometimes stated thus—*If a body can roll, it will roll, in preference to slipping.* Exactly the same considerations as before determine the possibility or impossibility of the rolling motion. Such a motion will always take place if it does not require the total resistance to make with the normal to the surface of contact an angle $> \lambda$.

For example, let us discuss the following problem:—

A heavy cubical block rests on a rough horizontal plane, and a string attached to the middle of one of the upper edges passes over a smooth pulley, and sustains a weight which is gradually increased. Find the nature of the initial motion of the block, the string and the vertical through the centre of gravity of the block being in the same vertical plane.

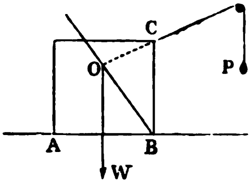

Fig. 198.

Let ABC (Fig. 198) be the vertical plane in which all the forces act; CO the line of the string, intersecting the vertical through the centre of gravity of the block in O; P the weight of the suspended mass, and W the weight of the block. (Since the length of the string is immaterial, no linear magnitude can enter into the result, therefore the side of the block need not be known.)

Now in all such cases as this, it is necessary to observe the following rules:—

1. Write down the motions of the system which are *geometrically* possible.

2. Exclude those which would obviously violate any of the fundamental rules of Statics.

3. If there remain possible cases of *slipping* and *rolling* (or *turning over*), solve the problem on the supposition that equilibrium is broken in the latter way, and if this does not require too great a value of the angle of friction, equilibrium will be broken in this way.

In the present case, the following motions are *geometrically* possible:—

(α) The block may be lifted vertically off the plane.

(β) It may turn round the edge A.

(γ) It may slide in the direction AB.

(δ) It may turn round the edge B.

Now (α) is obviously excluded, because if the block is just out of contact with the horizontal plane, it is acted on by only two forces, namely, its own weight and the tension of the string. But since these cannot be equal and opposite, equilibrium cannot be broken in this way.

Suppose (β) to happen. Then the total resistance of the plane passes through A and through O. But it is impossible that three forces acting in the directions of AO, OC, and OW could be in equilibrium. Hence (β) is excluded.

The cases (γ) and (δ) remain. Now in virtue of the principle, if (δ) is possible, it will happen. Solve, then, on the supposition that the block turns round B. It is then kept in equilibrium by its weight, the tension, and the total resistance which must act in BO. If the $\angle CBO$ is less than λ, the angle of friction, the block will turn round B; but if $CBO > \lambda$, this motion is impossible, and slipping must take place in the direction AB.

To express this analytically, let θ be the angle made with the horizon by the string OC, and let fall from O a perpendicular on BC meeting BC in p. Then

$$\tan CBO = \frac{Op}{Bp} = \frac{Op}{BC - Op \,.\, \tan\theta} = \frac{1}{2 - \tan\theta}.$$

Hence if μ (or $\tan\lambda$) be $> \dfrac{1}{2 - \tan\theta}$, the block can turn round B, and will do so if P is gradually increased.

The magnitude of P which will just cause the tilting of the block is found by taking moments about B. We evidently obtain

$$P = \tfrac{1}{2} W \sec\theta.$$

Suppose that $CBO > \lambda$, or that $\mu < \dfrac{1}{2 - \tan\theta}$. Then the increase of P will produce a sliding motion, and we can easily find the magnitude and point of application of the total resistance of the plane. Now since $CBO > \lambda$, the point, M, of application of the total resistance of the plane, is found by drawing from O a line OM making with the normal to the plane an angle $= \lambda$. The point M lies between B and the point in which the vertical through O cuts AB. P can then be determined either by taking moments about M, or by resolving vertically and horizontally. Resolving vertically, we have,

$$R\cos\lambda = W - P\sin\theta;$$

resolving horizontally,

$$R\sin\lambda = P\cos\theta;$$

$$\therefore\quad \frac{P\cos\theta}{W - P\sin\theta} = \mu, \quad\text{or}\quad P = \frac{\mu W}{\cos\theta + \mu\sin\theta}.$$

The direction of the string might be so modified as to render possible either a sliding in the direction BA or a tilting over A. Thus, in Fig. 199, if the line of the string intersect the line of action of the weight in a point, O, below the horizontal plane, the two motions possible are evidently one of slipping in the direction AB and one of tilting over the edge A. The latter will take place if it can. If it does, the total resistance must act in the line OA, and for this the angle DAR must be $< \lambda$. But if DAR is $> \lambda$, the block will slip in the direction AB, since the horizontal component of the tension acts in this sense. The condition for tilting over A is now evidently

$$\mu > \frac{1}{\tan\theta - 2}.$$

The values of P corresponding to both kinds of motion are calculated as before.

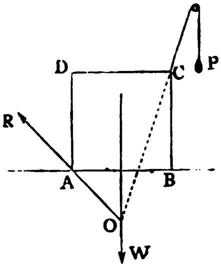

Fig. 199.

151.] **Limiting Positions of Equilibrium.** When a body rests in contact with any number of rough surfaces at several points, the equilibrium is said to be *limiting* if a slight alteration of a definite kind in the circumstances of the body would cause the equilibrium to be broken. The slight alteration referred to depends on the nature of the particular problem of equilibrium. As has been explained in p. 72, every statical problem relating to the equilibrium of a body is always one or other of the three following :—

(*a*) What is the least force that will sustain a body in a given position on given surfaces, or the greatest force that will allow it to rest in such a position ?

(*b*) With given forces and given supporting surfaces, what is the position of equilibrium such that if this position be slightly altered, the body will not rest ?

(*c*) With given forces, what is the least amount of roughness of the surface or surfaces which will allow the body to rest in a given position?

Thus in Fig. 198 of the last Article, supposing that the angle $CBO < \lambda$, the equilibrium of the block will be limiting if $P = \frac{1}{2} W \sec \theta$; for if P is slightly increased above this value, the block will turn over B.

Again, in Fig. 197 of the same Article, supposing the question to relate to the position of equilibrium, the beam AB will be in limiting equilibrium if its inclination to the horizon be $= \tan^{-1} \left(\frac{1-\mu^2}{2\mu}\right)$, because if it be slightly lowered below this position, it will slip.

Finally, if in the same figure we wish the beam to be sustained at any inclination a to the horizon between the equally rough vertical and horizontal planes, the equilibrium will be limiting if the angle of friction $= \frac{\pi}{4} - \frac{a}{2}$, because, if it be less than this, the beam will slip.

152.] **Comparative Safety of Equilibrium of a System at different Points.** When in a system in equilibrium the directions of the total resistances at the various points of contact with rough surfaces are known, we are enabled to say at which of the points slipping is most likely to happen in case some of the circumstances should be altered.

This will be rendered clear by the following examples, taken from Jellett's "Theory of Friction," p. 61 :—

Two uniform beams, AC and BC, connected at C by a smooth hinge, are placed, in a vertical plane, with their lower extremities, A and B, resting on a rough horizontal plane. If equilibrium be on the point of being broken, determine how this will happen.

Fig. 170, example 4, p. 228, will represent the beams if the hinges at A and B are conceived to be removed and these points rest on the ground. Then, exactly as in that example, the direction of the mutual resistance at C is determined. Supposing AC to be the longer beam, it is clear that the angle which the total resistance, AQ, at A makes with the normal to the surface of contact (i.e., to the ground) is greater than the angle which the total resistance BQ makes with the normal at B.

For
$$\frac{\tan AQn}{\tan BQn} = \frac{An}{Bn}.$$

Now $An = Am + mn$; and if $2a$, $2b$, $2c$, are the sides BC, CA, AB, we have

$$Am = b\cos\alpha,\ mn = fG = \frac{ac}{a+b} = \frac{a(b\cos\alpha + a\cos\beta)}{a+b},$$

$$\therefore\ An = b\cos\alpha + \frac{a(b\cos\alpha + a\cos\beta)}{a+b}$$

$$= \frac{(b^2 + 2ab)\cos\alpha + a^2\cos\beta}{a+b}.$$

Similarly
$$Bn = \frac{(a^2 + 2ab)\cos\beta + b^2\cos\alpha}{a+b};$$

therefore
$$An - Bn = \frac{2ab}{a+b}(\cos\alpha - \cos\beta).$$

But since $AC > BC$, $\cos\alpha > \cos\beta$, therefore $An > Bn$. Hence the angle $AQn > BQn$; that is, the total resistance at A makes with the normal at A an angle greater than that made by the total resistance at B with the normal at B. Consequently, if any circumstance should continually diminish the angle of friction (which is supposed to be the same for both beams) the total resistance at A would be the first to attain its limiting obliquity to the normal, and slipping would then take place at A in the direction BA, while the beam BC would turn round B.

We might inquire which of the beams will first slip if they are drawn out so as to increase the angle C, and the same result will follow, since for any given position of the beams the directions of all the resistances are determinate. In each case the angle AQn must be the first to reach the value λ, and therefore the longer beam, AC, must slip first.

The result may also be expressed thus—in any given position of rest, equilibrium is more safe at B than at A.

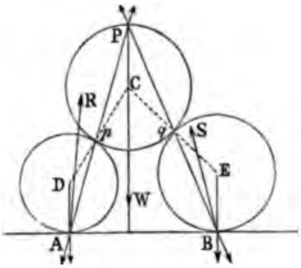

Fig. 200.

There are also cases in which the comparative safety of equilibrium can be determined, although the directions of total resistance are not completely determinate at *all* the points at contact. For example—two unequal cylinders rest on the ground at given points, A

and B (Fig. 200), while a third cylinder rests on them at points p and q.

Supposing either that there is a gradual diminution of the coefficient of friction (which is the same at all the points of contact), or that the lower cylinders are gradually drawn asunder, determine the nature of the initial motion of the system.

Denote the cylinders by the letters at their centres. Then the cylinder D is kept in equilibrium by three forces—namely, 1st, its weight, which acts through A; 2nd, the total resistance of the ground, which also acts through A; and 3rd, the total resistance of the cylinder C at p. Now, since the first two forces act through A, the third must also pass through this point. Hence the total resistance at p acts in the line pA, and therefore the total resistance of the ground at A must take some intermediate (but unknown) direction, AR. In the same way, the total resistance at q is proved to act in the line qB, and the total resistance of the ground at B must act in some direction, BS, intermediate to BE and Bq. The resistances in Ap and Bq at p and q meet in a point, P, on the circumference of the upper cylinder.

Now the comparative safety of equilibrium at the different points of contact, A, B, p, q, will depend on the angles made by the total resistances at these points with the normals to the surfaces of contact; and it is manifest that since the angle $DAp > DAR$ and $DpA = DAp$, the total resistance at p makes a greater angle with the normal, DC, to the surface of contact than that which the total resistance at A makes with the normal AD. Hence equilibrium is safer at A than at p. For a similar reason, equilibrium is safer at B than at q. Consequently the final comparison is to be made between the points p and q. Now the line pq can be proved by geometry to pass through the point in which ED intersects BA; and supposing the radius $BE > AD$, this point will be at the left-hand side of the figure. Let a be the acute angle which pq makes with the ground. Then, since in the triangle pCq the base angles at p and q are equal, it is easy to see that $\angle qCW - \angle pCW = 2a$, or $qCW > pCW$. But the angle which the total resistance at q makes with the normal qC is $\frac{1}{2}\, qCW$, and the angle which the total resistance at p makes with the normal pC is $\frac{1}{2} pCW$; therefore if the friction were gradually and uniformly diminished everywhere, or the cylinders drawn out, the resistance at q would reach its limiting obliquity before that at p. Hence the initial motion will be a slipping of the cylinders C and E at the point q, and a motion of rotation at the other points of contact.

153.] **Virtual Work of the Total Resistance.** Suppose one rough body to roll on another fixed one through a small angle whose magnitude is regarded as an infinitesimal of the first order. Then, neglecting infinitesimals of a higher order, the point of the rolling surface in contact with the other surface is at rest during the displacement—that is, the virtual displacement of the point of application of the total resistance between the two bodies is zero. Hence for a virtual displacement which consists of a small rolling motion of one rough body on another, the total resistance will not enter into the equation of virtual work of *either* body. Of course in no case can the mutual action of two rigid bodies in contact enter into an equation of virtual work for *both* bodies.

It is a principle in Kinetics that in a motion of pure rolling of a body on a rough fixed surface no work is done between any two positions by the total resistance—a principle which the student will have no difficulty in comprehending, since for each small motion the work done by this force is infinitesimal compared with the work done by other forces acting on the body.

154.] **Friction as dependent on Initial Arrangements.** In dealing with natural solids, and not with strictly rigid or indeformable bodies, the existence or non-existence of friction sometimes depends on the way in which a body or system has been placed in the position which we are considering. This will be made clear by the following example. A heavy trap door (or a bar), AB, Fig. 132, p. 166, moveable about a fixed horizontal axis at A, has a rope attached at B, and this rope is also attached to any fixed point C; determine the pressure on the axis A.

The line of action of the pressure must, of course, go through O, the point of meeting of the other two forces, but beyond this we know nothing about it until we know the nature of the axis. If the axis is smooth, or if it is rough but so worn that the contact of the door with it takes place along a single line, the action between the door and the axis will consist of a force passing through the axis, as has been amply explained in Art. 106. But if the axis is rough and contact takes place all round it, the

line of action of the resultant force is not generally determinate. However, even in this case this resultant force may pass through the axis. The axis being rough, let us imagine it to become smooth, and what motion results? The rope, being slightly extensible, would yield a little, and slipping would take place over a small surface at the axis; so that the supposition of smoothness alters the circumstances of the case. But suppose that (the axis being still rough) the rope has been stretched, when the door is placed in position, to such an extent that the moment of its tension about the axis is equal to the moment of the weight of the door; then clearly if we imagine the axis to become smooth, no motion will result—no slipping at the axis; and since the displacement which friction is required to prevent does not take place, friction does not act, and the case is the same as if the axis were smooth. The resultant in this case is therefore determinate.

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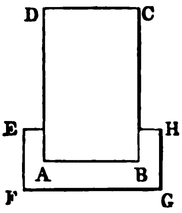
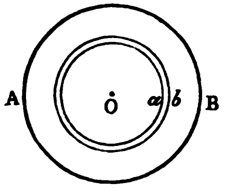


Fig. 201.



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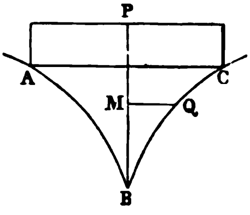

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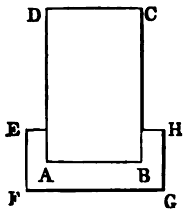

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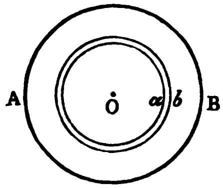

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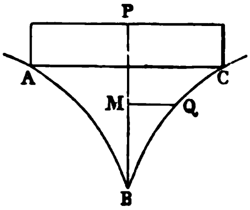

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and B (Fig. 200), while a third cylinder rests on them at points p and q.

Supposing either that there is a gradual diminution of the coefficient of friction (which is the same at all the points of contact), or that the lower cylinders are gradually drawn asunder, determine the nature of the initial motion of the system.

Denote the cylinders by the letters at their centres. Then the cylinder D is kept in equilibrium by three forces—namely, 1st, its weight, which acts through A; 2nd, the total resistance of the ground, which also acts through A; and 3rd, the tota resistance of the cylinder C at p. Now, since the first two forc act through A, the third must also pass through this poir Hence the total resistance at p acts in the line pA, and therefc the total resistance of the ground at A must take some int mediate (but unknown) direction, AR. In the same way, total resistance at q is proved to act in the line qB, and the t resistance of the ground at B must act in some direction, intermediate to BE and Bq. The resistances in Ap and I p and q meet in a point, P, on the circumference of the cylinder.

Now the comparative safety of equilibrium at the di points of contact, A, B, p, q, will depend on the angles m the total resistances at these points with the normals surfaces of contact; and it is manifest that since the $DAp > DAR$ and $DpA = DAp$, the total resistance at p greater angle with the normal, DC, to the surface of than that which the total resistance at A makes with th AD. Hence equilibrium is safer at A than at p. For reason, equilibrium is safer at B than at q. Consequ final comparison is to be made between the points p and the line pq can be proved by geometry to pass through in which ED intersects BA; and supposing the radius this point will be at the left-hand side of the figure. the acute angle which pq makes with the ground. Th the triangle pCq the base angles at p and q are equal, see that $\angle qCW - \angle pCW = 2a$, or $qCW > pCW$. Br which the total resistance at q makes with the normal and the angle which the total resistance at p ma normal pC is $\frac{1}{2}pCW$; therefore if the friction were f uniformly diminished everywhere, or the cylinders r

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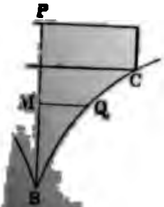

Fig. 103.

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line of action of the resultant force is not generally determinate. However, even in this case this resultant force may pass through the axis. The axis being rough, let us imagine it to become smooth, and what motion results? The rope, being slightly extensible, would yield a little, and slipping would take place over a small surface at the axis; so that the supposition of smoothness alters the circumstances of the case. But suppose that (the axis being still rough) the rope has been stretched, when the door is placed in position, to such an extent that the moment of its tension about the axis is equal to the moment of the weight of the door; then clearly if we imagine the axis to become smooth, no motion will result—no slipping at the axis; and since the displacement which friction is required to prevent does not take place, friction does not act, and the case is the same as if the axis were smooth. The resultant in this case is therefore determinate.

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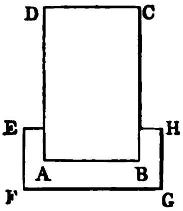

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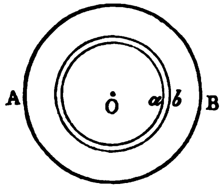

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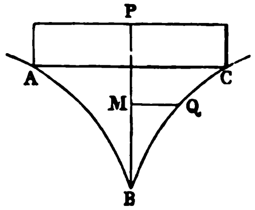

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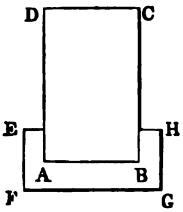

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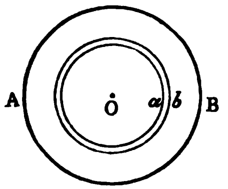

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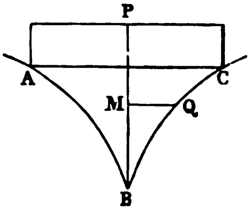

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so that tan λ is the greatest ratio of the force of friction to the normal pressure. This quantity we have called μ.

If a rigid rod, M (p. 52), of negligible weight, be pressed against a rough surface at O, the greatest angle that the rod can make with the normal is the angle of friction. For, since the rod is acted on by only two forces, viz., the applied pressure and the total resistance at O, these must be equal and opposite, that is along the rod. Hence the greatest obliquity of the rod to the normal is λ.

If the resistance to slipping is not the same in different azimuths, i.e., if it is different in different planes through the normal, the value of λ will not be the same in all these planes, and the cone of friction will not be a right circular cone.

149.] **Axiomatic Law of Friction.** We have said that the total resistance of a rigid surface is a force which can assume any magnitude. This force will in any given case be exerted by the surface to such an extent as is necessary to preserve equilibrium, but to no greater extent. It is in its nature a *passive resistance*, i.e., one which can be exerted to any extent, but which will not be exerted beyond the bare requirements of the case. Within certain limits, also, it can assume any direction, and in any given case it will, if possible, assume such a direction as will preserve equilibrium. In fact, in virtue of its *passive* nature, we must regard the resistance of a rough surface as an opposition called into existence by the action of external forces; and it is evident that these forces will call into play only that amount of opposing force, exact both in magnitude and in direction, which will just counteract their own action.

The amount of assumption contained in this principle is enunciated in the following axiom :—

The total resistance which acts at any point of a rough surface will, if possible, assume such a magnitude and direction as will preserve equilibrium at that point.

This axiom is sometimes expressed thus :—*If passive resistances can give equilibrium, they will.*

150.] **Remarks on this Axiom.** Two important observations must be made on the principles contained in this axiom. Firstly, it is important to understand the circumstances which may render it impossible for the resistances of rough surfaces to preserve the equilibrium of a system in any given position.

Suppose that a body, acted on by given external forces, is in contact with a rough surface at a single point, *P*. Then, for equilibrium, it is necessary that the resultant of the given external forces should pass through *P*, and that the total resistance at *P* should be equal and opposite to this resultant. But if the direction of the resultant makes with the normal to the surface of contact at *P* an angle $> \lambda$, it is impossible that the total resistance could take the required direction, and equilibrium cannot subsist.

Again, take the case in which a heavy beam, *AB* (Fig. 195), rests against a rough horizontal and an equally rough vertical plane. Describe round the normals to the planes at *A* and *B* the cones of friction, and let the sections of these cones by the plane of the figure be *rAs* and *pBs*. Let *G* be the centre of gravity of the beam, and *GV* the vertical line through it.

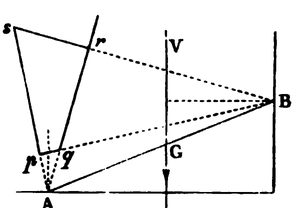

Fig. 195.

Then the beam, if in equilibrium, is so under the action of three forces, namely, the weight through *G* and the total resistances at *A* and *B*. These three forces must meet in a point, and if it be possible to find a point in which they can meet, the resistances will assume proper values. Now, in the figure it is impossible to find any point on *GV*, the line of action of the weight, the lines drawn from which to *A* and *B* could be directions of possible resistance at *both A* and *B*. For the portion of *GV* which is inside the cone of friction at *B* is outside the cone of friction at *A*, and *vice versâ*. Hence, for equilibrium, *there must be some portion of the line GV included in the space pqrs, common to both cones of friction.*

Unless this condition is satisfied, it is not possible for the total resistances to give equilibrium, whatever their magnitudes may be. A possible position of equilibrium is represented in Fig. 196. For, if from any point on the portion, *mn*, of *GV*, which is included in the space common to both cones of friction, lines be drawn to *A* and *B*, these lines are possible directions of total resistance at *A* and *B*; and in this case the actual magnitudes and directions of the resistances at *A* and *B* cannot be determined by what is called Rational Statics.

If it be proposed to find the position of *limiting* equilibrium, that is, the position in which the beam is bordering on motion,

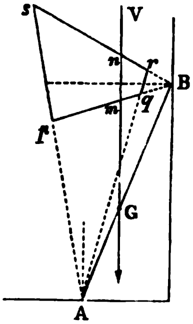

Fig. 196.

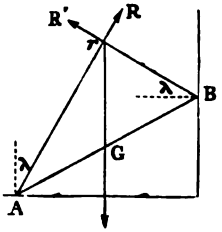

Fig. 197.

we must make the vertical through G pass through r, as in Fig. 197.

In this case there is only one point on GV which is inside both cones of friction, viz., the point r. Hence the total resistances act in rA and rB, and each makes the limiting angle (λ) with the corresponding normal. Moreover, both resistances are now determinate. If θ be the angle made by the beam with the horizon, we have, from the triangle ArB,

$$2 \cot rGB = \cot ArG - \cot BrG,$$

or

$$2 \tan \theta = \cot \lambda - \tan \lambda,$$

$$= \frac{1-\mu^2}{\mu};$$

which defines the position of limiting equilibrium.

It may, therefore, in certain cases be impossible for the total resistance at one or more points to preserve equilibrium; and this impossibility is always due to something in the arrangement of the figure or the external forces which requires the direction of the resistance to make with the normal to the surface of contact an angle > the angle of friction.

Again, in the axiom is contained the following important proposition:—

If a body rests against a rough surface at a point, and if the equilibrium is about to be broken by some change in the acting forces, equilibrium at that point will, if possible, be broken by a rolling instead of a sliding motion.

For, in this case, the point of the body actually in contact with the surface would be kept at rest. This part of the axiom is sometimes stated thus—*If a body can roll, it will roll, in preference to slipping.* Exactly the same considerations as before determine the possibility or impossibility of the rolling motion. Such a motion will always take place if it does not require the total resistance to make with the normal to the surface of contact an angle $> \lambda$.

For example, let us discuss the following problem:—

A heavy cubical block rests on a rough horizontal plane, and a string attached to the middle of one of the upper edges passes over a smooth pulley, and sustains a weight which is gradually increased. Find the nature of the initial motion of the block, the string and the vertical through the centre of gravity of the block being in the same vertical plane.

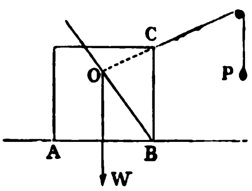

Fig. 198.

Let ABC (Fig. 198) be the vertical plane in which all the forces act; CO the line of the string, intersecting the vertical through the centre of gravity of the block in O; P the weight of the suspended mass, and W the weight of the block. (Since the length of the string is immaterial, no linear magnitude can enter into the result, therefore the side of the block need not be known.)

Now in all such cases as this, it is necessary to observe the following rules:—

1. Write down the motions of the system which are *geometrically* possible.

2. Exclude those which would obviously violate any of the fundamental rules of Statics.

3. If there remain possible cases of *slipping* and *rolling* (or *turning over*), solve the problem on the supposition that equilibrium is broken in the latter way, and if this does not require too great a value of the angle of friction, equilibrium will be broken in this way.

In the present case, the following motions are *geometrically* possible:—

(a) The block may be lifted vertically off the plane.

(β) It may turn round the edge A.

(γ) It may slide in the direction AB.

(δ) It may turn round the edge B.

Now (α) is obviously excluded, because if the block is just out of contact with the horizontal plane, it is acted on by only two forces, namely, its own weight and the tension of the string. But since these cannot be equal and opposite, equilibrium cannot be broken in this way.

Suppose (β) to happen. Then the total resistance of the plane passes through A and through O. But it is impossible that three forces acting in the directions of AO, OC, and OW could be in equilibrium. Hence (β) is excluded.

The cases (γ) and (δ) remain. Now in virtue of the principle, if (δ) is possible, it will happen. Solve, then, on the supposition that the block turns round B. It is then kept in equilibrium by its weight, the tension, and the total resistance which must act in BO. If the $\angle CBO$ is less than λ, the angle of friction, the block will turn round B; but if $CBO > \lambda$, this motion is impossible, and slipping must take place in the direction AB.

To express this analytically, let θ be the angle made with the horizon by the string OC, and let fall from O a perpendicular on BC meeting BC in p. Then

$$\tan CBO = \frac{Op}{Bp} = \frac{Op}{BC - Op \,.\, \tan\theta} = \frac{1}{2 - \tan\theta}.$$

Hence if μ (or $\tan\lambda$) be $> \frac{1}{2 - \tan\theta}$, the block can turn round B, and will do so if P is gradually increased.

The magnitude of P which will just cause the tilting of the block is found by taking moments about B. We evidently obtain

$$P = \tfrac{1}{2} W \sec\theta.$$

Suppose that $CBO > \lambda$, or that $\mu < \frac{1}{2 - \tan\theta}$. Then the increase of P will produce a sliding motion, and we can easily find the magnitude and point of application of the total resistance of the plane. Now since $CBO > \lambda$, the point, M, of application of the total resistance of the plane, is found by drawing from O a line OM making with the normal to the plane an angle $= \lambda$. The point M lies between B and the point in which the vertical through O cuts AB. P can then be determined either by taking moments about M, or by resolving vertically and horizontally. Resolving vertically, we have,

$$R\cos\lambda = W - P\sin\theta;$$

resolving horizontally,

$$R\sin\lambda = P\cos\theta;$$

$$\therefore\quad \frac{P\cos\theta}{W-P\sin\theta} = \mu, \quad\text{or}\quad P = \frac{\mu W}{\cos\theta+\mu\sin\theta}.$$

The direction of the string might be so modified as to render possible either a sliding in the direction BA or a tilting over A. Thus, in Fig. 199, if the line of the string intersect the line of action of the weight in a point, O, below the horizontal plane, the two motions possible are evidently one of slipping in the direction AB and one of tilting over the edge A. The latter will take place if it can. If it does, the total resistance must act in the line OA, and for this the angle DAR must be $< \lambda$. But if DAR is $> \lambda$, the block will slip in the direction AB, since the horizontal component of the tension acts in this sense. The condition for tilting over A is now evidently

$$\mu > \frac{1}{\tan\theta - 2}.$$

The values of P corresponding to both kinds of motion are calculated as before.

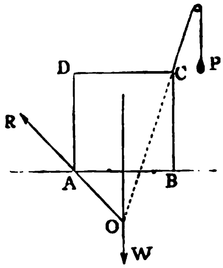

Fig. 199.

151.] **Limiting Positions of Equilibrium.** When a body rests in contact with any number of rough surfaces at several points, the equilibrium is said to be *limiting* if a slight alteration of a definite kind in the circumstances of the body would cause the equilibrium to be broken. The slight alteration referred to depends on the nature of the particular problem of equilibrium. As has been explained in p. 72, every statical problem relating to the equilibrium of a body is always one or other of the three following:—

(*a*) What is the least force that will sustain a body in a given position on given surfaces, or the greatest force that will allow it to rest in such a position?

(*b*) With given forces and given supporting surfaces, what is the position of equilibrium such that if this position be slightly altered, the body will not rest?

(*c*) With given forces, what is the least amount of roughness of the surface or surfaces which will allow the body to rest in a given position?

Thus in Fig. 198 of the last Article, supposing that the angle $CBO < \lambda$, the equilibrium of the block will be limiting if $P = \frac{1}{2} W \sec\theta$; for if P is slightly increased above this value, the block will turn over B.

Again, in Fig. 197 of the same Article, supposing the question to relate to the position of equilibrium, the beam AB will be in limiting equilibrium if its inclination to the horizon be $= \tan^{-1}\left(\frac{1-\mu^2}{2\mu}\right)$, because if it be slightly lowered below this position, it will slip.

Finally, if in the same figure we wish the beam to be sustained at any inclination a to the horizon between the equally rough vertical and horizontal planes, the equilibrium will be limiting if the angle of friction $= \frac{\pi}{4} - \frac{a}{2}$, because, if it be less than this, the beam will slip.

152.] **Comparative Safety of Equilibrium of a System at different Points.** When in a system in equilibrium the directions of the total resistances at the various points of contact with rough surfaces are known, we are enabled to say at which of the points slipping is most likely to happen in case some of the circumstances should be altered.

This will be rendered clear by the following examples, taken from Jellett's "Theory of Friction," p. 61:—

Two uniform beams, AC and BC, connected at C by a smooth hinge, are placed, in a vertical plane, with their lower extremities, A and B, resting on a rough horizontal plane. If equilibrium be on the point of being broken, determine how this will happen.

Fig. 170, example 4, p. 228, will represent the beams if the hinges at A and B are conceived to be removed and these points rest on the ground. Then, exactly as in that example, the direction of the mutual resistance at C is determined. Supposing AC to be the longer beam, it is clear that the angle which the total resistance, AQ, at A makes with the normal to the surface of contact (i.e., to the ground) is greater than the angle which the total resistance BQ makes with the normal at B.

For
$$\frac{\tan AQn}{\tan BQn} = \frac{An}{Bn}.$$

Now $An = Am + mn$; and if $2a$, $2b$, $2c$, are the sides BC, CA, AB, we have

$$Am = b\cos\alpha,\ mn = fG = \frac{ac}{a+b} = \frac{a(b\cos\alpha + a\cos\beta)}{a+b},$$

$$\therefore\ An = b\cos\alpha + \frac{a(b\cos\alpha + a\cos\beta)}{a+b}$$

$$= \frac{(b^2 + 2ab)\cos\alpha + a^2\cos\beta}{a+b}.$$

Similarly
$$Bn = \frac{(a^2 + 2ab)\cos\beta + b^2\cos\alpha}{a+b};$$

therefore
$$An - Bn = \frac{2ab}{a+b}(\cos\alpha - \cos\beta).$$

But since $AC > BC$, $\cos\alpha > \cos\beta$, therefore $An > Bn$. Hence the angle $AQn > BQn$; that is, the total resistance at A makes with the normal at A an angle greater than that made by the total resistance at B with the normal at B. Consequently, if any circumstance should continually diminish the angle of friction (which is supposed to be the same for both beams) the total resistance at A would be the first to attain its limiting obliquity to the normal, and slipping would then take place at A in the direction BA, while the beam BC would turn round B.

We might inquire which of the beams will first slip if they are drawn out so as to increase the angle C, and the same result will follow, since for any given position of the beams the directions of all the resistances are determinate. In each case the angle AQn must be the first to reach the value λ, and therefore the longer beam, AC, must slip first.

The result may also be expressed thus—in any given position of rest, equilibrium is more safe at B than at A.

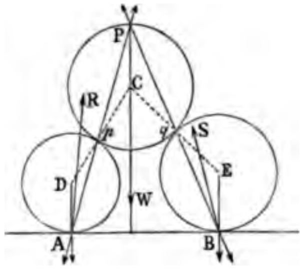

Fig. 200.

There are also cases in which the comparative safety of equilibrium can be determined, although the directions of total resistance are not completely determinate at *all* the points at contact. For example—two unequal cylinders rest on the ground at given points, A

and B (Fig. 200), while a third cylinder rests on them at points p and q.

Supposing either that there is a gradual diminution of the coefficient of friction (which is the same at all the points of contact), or that the lower cylinders are gradually drawn asunder, determine the nature of the initial motion of the system.

Denote the cylinders by the letters at their centres. Then the cylinder D is kept in equilibrium by three forces—namely, 1st, its weight, which acts through A; 2nd, the total resistance of the ground, which also acts through A; and 3rd, the total resistance of the cylinder C at p. Now, since the first two forces act through A, the third must also pass through this point. Hence the total resistance at p acts in the line pA, and therefore the total resistance of the ground at A must take some intermediate (but unknown) direction, AR. In the same way, the total resistance at q is proved to act in the line qB, and the total resistance of the ground at B must act in some direction, BS, intermediate to BE and Bq. The resistances in Ap and Bq at p and q meet in a point, P, on the circumference of the upper cylinder.

Now the comparative safety of equilibrium at the different points of contact, A, B, p, q, will depend on the angles made by the total resistances at these points with the normals to the surfaces of contact; and it is manifest that since the angle $DAp > DAR$ and $DpA = DAp$, the total resistance at p makes a greater angle with the normal, DC, to the surface of contact than that which the total resistance at A makes with the normal AD. Hence equilibrium is safer at A than at p. For a similar reason, equilibrium is safer at B than at q. Consequently the final comparison is to be made between the points p and q. Now the line pq can be proved by geometry to pass through the point in which ED intersects BA; and supposing the radius $BE > AD$. this point will be at the left-hand side of the figure. Let α be the acute angle which pq makes with the ground. Then, since in the triangle pCq the base angles at p and q are equal, it is easy to see that $\angle qCW - \angle pCW = 2\alpha$, or $qCW > pCW$. But the angle which the total resistance at q makes with the normal qC is $\frac{1}{2}\, qCW$, and the angle which the total resistance at p makes with the normal pC is $\frac{1}{2} pCW$; therefore if the friction were gradually and uniformly diminished everywhere, or the cylinders drawn out, the resistance at q would reach its limiting obliquity before that at p. Hence the initial motion will be a slipping of the cylinders C and E at the point q, and a motion of rotation at the other points of contact.

153.] **Virtual Work of the Total Resistance.** Suppose one rough body to roll on another fixed one through a small angle whose magnitude is regarded as an infinitesimal of the first order. Then, neglecting infinitesimals of a higher order, the point of the rolling surface in contact with the other surface is at rest during the displacement—that is, the virtual displacement of the point of application of the total resistance between the two bodies is zero. Hence for a virtual displacement which consists of a small rolling motion of one rough body on another, the total resistance will not enter into the equation of virtual work of *either* body. Of course in no case can the mutual action of two rigid bodies in contact enter into an equation of virtual work for *both* bodies.

It is a principle in Kinetics that in a motion of pure rolling of a body on a rough fixed surface no work is done between any two positions by the total resistance—a principle which the student will have no difficulty in comprehending, since for each small motion the work done by this force is infinitesimal compared with the work done by other forces acting on the body.

154.] **Friction as dependent on Initial Arrangements.** In dealing with natural solids, and not with strictly rigid or indeformable bodies, the existence or non-existence of friction sometimes depends on the way in which a body or system has been placed in the position which we are considering. This will be made clear by the following example. A heavy trap door (or a bar), AB, Fig. 132, p. 166, moveable about a fixed horizontal axis at A, has a rope attached at B, and this rope is also attached to any fixed point C; determine the pressure on the axis A.

The line of action of the pressure must, of course, go through O, the point of meeting of the other two forces, but beyond this we know nothing about it until we know the nature of the axis. If the axis is smooth, or if it is rough but so worn that the contact of the door with it takes place along a single line, the action between the door and the axis will consist of a force passing through the axis, as has been amply explained in Art. 106. But if the axis is rough and contact takes place all round it, the

line of action of the resultant force is not generally determinate. However, even in this case this resultant force may pass through the axis. The axis being rough, let us imagine it to become smooth, and what motion results? The rope, being slightly extensible, would yield a little, and slipping would take place over a small surface at the axis; so that the supposition of smoothness alters the circumstances of the case. But suppose that (the axis being still rough) the rope has been stretched, when the door is placed in position, to such an extent that the moment of its tension about the axis is equal to the moment of the weight of the door; then clearly if we imagine the axis to become smooth, no motion will result—no slipping at the axis; and since the displacement which friction is required to prevent does not take place, friction does not act, and the case is the same as if the axis were smooth. The resultant in this case is therefore determinate.

155.] **Friction of a Pivot.** Let a cylindrical pivot, *ABCD* (Fig. 201), on the top of which a given force is applied, revolve in a closely fitting bearing, *EFGH*, and let it be required to calculate the moment of the friction on the base, *AB*, about the axis of the pivot. Suppose Fig. 202 to represent the base of the pivot, and let $P =$ the whole normal pressure on the base, which we shall suppose to be uniformly distributed over the base. Divide the area AB into a number of narrow circular strips, of which one is represented in the Figure. Let $Oa = x$, $Ob = x + dx$, $OB = r$, $\mu =$ coefficient of friction. Then since the whole pressure is uniformly distributed, the pressure on the strip whose area is $= 2\pi x dx$ is $\frac{P}{\pi r^2}\cdot 2\pi x dx$, or $\frac{2Px\,dx}{r^2}\cdot$ Hence the sum of the forces of friction, acting in the directions of the tangents to the

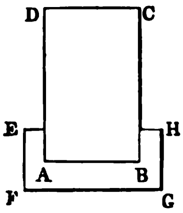

Fig. 201.

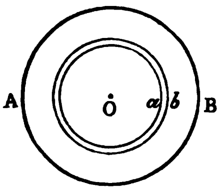

Fig. 202.

strip, is $\frac{2\mu P x\, dx}{r^2}$. But since the tangents to the strip are all at the same distance from the centre, the moment of friction on the strip is equal to the sum of the forces of friction multiplied by the radius, x, of the strip. Hence the moment of friction over the whole surface is

$$\int_0^r \frac{2\mu P x^2 dx}{r^2}, \quad \text{or} \quad \frac{2}{3}\mu P r. \tag{1}$$

If the base, instead of being a full circle, is a ring, or *collar*, whose internal and external radii are r_1 and r_2, the friction per unit of surface is $\frac{\mu P}{\pi(r_2^2 - r_1^2)}$, and the moment of friction is

$$\int_{r_1}^{r_2} \frac{2\mu P x^2 dx}{(r_2^2 - r_1^2)}, \quad \text{or} \quad \frac{2}{3}\mu P . \frac{r_2^3 - r_1^3}{r_2^2 - r_1^2}. \tag{2}$$

156.] **Wearing away of the Step.** The piece which supports a pivot, and in which it revolves, is called a *step*. When the pivot revolves, the friction against the step wears away its own surface and that of the step. The amount of wear at any point of the step depends on the magnitude of the force of friction and the relative velocity of the rubbing surfaces at this point. Thus, suppose that ABC (Fig. 203) represents a section of the step through the axis, BP, of the pivot, and that Q is any point of contact of the pivot and step. If f is the magnitude of the force of friction at Q, the wearing at Q in the direction of the normal will be proportional to f and also to the amount of rubbing surface which passes over Q in a unit of time. Supposing the pivot to revolve round its axis with an angular velocity ω, the point of the pivot in contact with Q moves in a horizontal circle with a velocity $= \omega \,.\, QM$, or $\omega \,.\, y$; QM, or y, being the perpendicular from Q on the axis of the pivot.

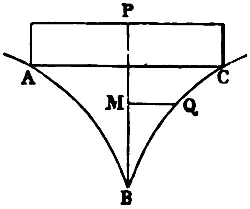

Fig. 203.

But the amount of rubbing surface which passes over Q in a unit of time is evidently proportional to the velocity at Q. Hence the normal wearing of the surface at Q is proportional to

$$\omega f y.$$

If n be the magnitude of the normal pressure per unit of surface at Q, and μ the coefficient of friction, we have $f = \mu n$.

Hence the normal wearing of the surface at Q is proportional to

$$\omega \mu n y. \qquad (a)$$

157.] **Friction of a Conical Pivot.** Let ABC (Fig. 204) represent a section of a conical step by a plane through the axis, BP, of the pivot, APC being the surface at which the pivot enters the step.

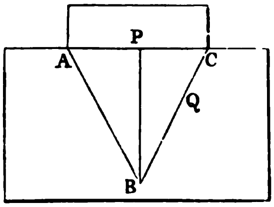

Fig. 204.

Supposing that the pressure on the top of the pivot is uniformly distributed, it will evidently be uniformly distributed over the area APC; that is, there will be a constant normal pressure, n, per unit of area on APC. Now it is impossible to determine the law of distribution of the pressure on the step. The following investigation proceeds on the assumption that the normal pressure per unit of area, or as it is properly called, the normal *intensity of pressure*, is constant over the surface of contact.

Let n be the constant pressure per unit of surface of the step.

If ds is a small element of the line BC at Q, the distance of which from BP is y, the corresponding elementary strip of conical surface is $2\pi y ds$, and the moment round BP of the friction on this strip is

$$2\pi y ds \times \mu n \times y,$$

or

$$2\mu\pi n y^2 ds.$$

Putting $ds = \frac{dy}{\sin\theta}$, and integrating over the surface of the step from $y = 0$ to $y = PC = r$, we have the moment of the whole friction equal to

$$\frac{2\mu\pi n r^3}{3\sin\theta}.$$

If $P =$ the whole pressure on the top of the pivot, $n = \frac{P}{\pi r^2}$; hence the moment of friction

$$= \frac{2\mu}{3\sin\theta} \cdot Pr.$$

Comparing this with the result in Art. 155, we see that the moment of friction in the case of a conical is greater than in the case of a cylindrical pivot of equal radius.

158.] **The Tractory, or Anti-Friction Curve.** In the case of a conical pivot the wearing away of the step is not uniform at all points. Hence after a sufficient time the pivot will not be in perfect contact with its step. If, however, the step has such a form that the vertical wear is the same at all points, the pivot will simply sink into the piece which supports it, and remain always in contact throughout its surface with the step.

We propose to investigate the form of the step in which the vertical wear will be the same at all points. Let Fig. 205 represent a section of the step through the axis of the pivot, and let CC' be the vertical wear at C, and QQ' the vertical wear at Q. Then $CC' = QQ'$, Q being any point on the curve BC. Hence the new curve, $BQ'C'$, is simply the old curve BQC moved through a vertical distance $CC' = QQ' = h$, suppose.

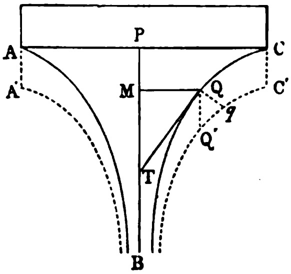

Fig. 205.

Now (Art. 156) the normal wear at Q per unit of surface is proportional to $\omega\mu n y$. Hence, if Qq is normal to the step at Q,

$$Qq = k\omega\mu n y,$$

n being the normal pressure per unit of surface on APC, which we also take to be the normal pressure per unit of surface on the step, and k a constant.

But $QQ' = \dfrac{Qq}{\cos Q'Qq} = \dfrac{Qq}{\sin MTQ}$, QT being the tangent to the curve at Q. Hence

$$h = \frac{k\omega\mu n y}{\dfrac{dy}{ds}},$$

or

$$y\frac{ds}{dy} = \frac{kh}{\omega\mu n} = \text{a constant},$$

or

$$QT = \text{a constant}.$$

Therefore the curve BC is such that the length of the tangent terminated by PB, or the axis of x, is constant at all points. This curve is known as the Tractory. If $t =$ the constant length of the tangent, and PC is the axis of y, we have

$$y\frac{ds}{dy} = t,$$

or

$$y\sqrt{1+\left(\frac{dx}{dy}\right)^2} = t,$$

or

$$\frac{\sqrt{t^2-y^2}}{y}\,dy = -\,dx,$$

the minus sign being given to the square root, because MQ diminishes as x increases. Integrating this last equation (by assuming $y = t\sin\phi$) we have for the equation of the tractory

$$t\log\frac{t-\sqrt{t^2-y^2}}{y} + x + \sqrt{t^2-y^2} = 0.$$

The curve approaches PB asymptotically, and the step is formed by the revolution of the curve round PB. This pivot is known as *Schiele's Anti-friction Pivot*

EXAMPLES*.

1. A uniform rectangular board, $ABCD$ (Fig. 206), rests in a vertical plane against two equally rough pegs, P and Q, in the same horizontal line, two adjacent sides of the board being each in contact with a peg. Find the position of limiting equilibrium.

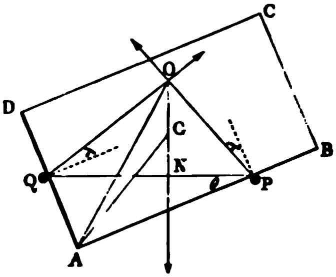

Fig. 206.

Let λ be the angle of friction, θ the inclination of the side AB to the horizon in the position of limiting equilibrium, G the centre of gravity of the board, $PQ = a$, and $AG = c$.

Then if the board is on the point of slipping down at Q and up at P, the total resistances at P and Q will act in the directions PO and QO, which are inclined at the angle λ to the normals at P and Q to the sides AB and AD, respectively. If O' (not represented in Figure) be the point of meeting of the normals at P and Q, it is clear that a circle will pass through the points $APO'OQ$; and therefore $\angle OAO' = \lambda$. And since $AO' = PQ = a$, we have

$$AO = a\cos\lambda. \qquad (1)$$

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* Many of the following examples are taken from Jellett's *Theory of Friction*.

Again, since $\angle O'QP = \theta$, we have $\angle QOG = \frac{\pi}{2} - (\lambda + \theta)$, and evidently, $\angle QOA = \theta$, therefore $\angle AOG = \frac{\pi}{2} - (\lambda + 2\theta)$. If $\angle GAB = a$, it is clear that $\angle AGN = \frac{\pi}{2} - (\theta + a)$. Now the position of equilibrium is found by the equation

$$AO \,.\, \sin AOG = AG \,.\, \sin AGN.$$

Substituting in this equation the value of AO from (1), we have

$$a \cos \lambda \,.\, \cos(\lambda + 2\theta) = c \,.\, \cos(a + \theta),$$

which defines the position of equilibrium.

2. A heavy uniform beam rests against a rough horizontal plane and against a rough vertical wall, the vertical plane through the beam being at right angles to the wall and the ground; determine the weight of the greatest mass that can be affixed to it at a given point, so that equilibrium may be preserved.

If the beam be inclined to the vertical at an angle less than the angle of friction for the beam and the ground, equilibrium cannot be broken by attaching a mass, however great, to any point of the beam.

Let AB (Fig. 207) be the beam, θ its inclination to the horizon, W its weight, $2a$ its length, P the weight of the mass suspended from the point Q in the beam, $BQ = x$, λ and λ' the angles of friction at A and B, respectively.

Draw the lines AO and BO, making the angles λ and λ' with the normals, An and Bm, at A and B.

Then when the resultant of W and P passes through O, equilibrium will be at its limit. For, if this resultant acts in a line to the left of OV, the vertical through O, it will be possible to find an infinite number of points on it such that when joined to A and B the joining lines will be possible directions of total resistance at A and B (see Art. 150).

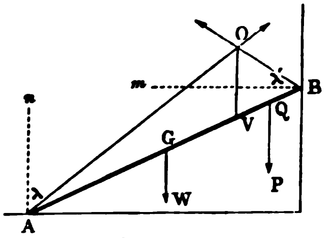

Fig. 207.

If the resultant of W and P acts in a line to the right of OV, there will be no point on it inside both cones of friction, and therefore equilibrium will be impossible. Hence for limiting equilibrium, we have by taking moments about O,

$$W \,.\, GV = P \,.\, QV,$$

G being the centre of gravity of the beam.

The lengths GV and QV are easily obtained from the data. We may observe that if the point Q lies between G and V, equilibrium can never be broken, however great P may be. For it will then be impossible by increasing P to bring the resultant of P and W to the right of OV.

These results follow also from the usual mode of solution of such a problem.

Let R and S be the normal reactions at A and B, and μ and μ' the coefficients of friction at these points. Then, resolving horizontally,

$$S = \mu R; \tag{2}$$

resolving vertically, $\quad R + \mu' S = P + W; \qquad (3)$

taking moments about B,

$$2aR(\cos\theta - \mu\sin\theta) = (Px + Wa)\cos\theta. \tag{4}$$

From (2) and (3) we have $\quad R = \dfrac{P+W}{1+\mu\mu'}$,

and by substituting this value of R in (4), we get

$$P = Wa\frac{1+\mu\mu' - 2(1-\mu\tan\theta)}{2a(1-\mu\tan\theta) - x(1+\mu\mu')}. \tag{5}$$

Now it is easy to see that $BO = 2a\dfrac{\cos(\theta+\lambda)}{\cos(\lambda-\lambda')}$, and $BV = BO \times \dfrac{\cos\lambda'}{\cos\theta}$; therefore $BV = 2a\dfrac{1-\mu\tan\theta}{1+\mu\mu'}$, and (5) may be written

$$P = W\cdot\frac{a - BV}{BV - x},$$

from which it appears that if $x = BV$, the required force is infinite; and if $x > BV$, it is negative, or equilibrium can never be broken by any downward force.

The second part of the problem follows from (5), because if $\mu\tan\theta > 1$, or, in other words, if the angle $nAB < \lambda$, the denominator will be negative. That it is impossible to break equilibrium in this case is evident from Fig. 208. For the point O is now at the right of the vertical wall, and at whatever point along AB the resultant of P and W acts, it is possible to find points on it which are within both cones of friction.

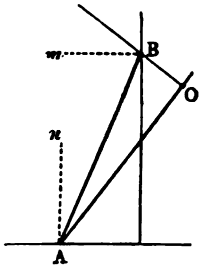

Fig. 208.

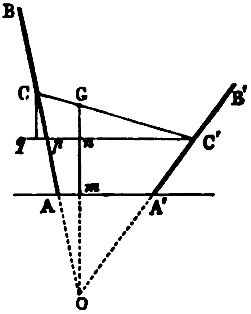

Fig. 209.

3. Two unequal uniform beams, connected by a light rope attached to their middle points, rest in a vertical plane, an extremity of each

beam resting on a rough horizontal plane. If the coefficient of friction is gradually diminished, which beam will slip first?

Let the beams be AB and $A'B'$ (Fig. 209), and let C and C' be their centres, and $AB > A'B'$. Now the beam AB is in equilibrium under the influence of three forces, viz., its weight, the tension of the rope CC', and the total resistance at A; and since the first two meet in C, the third must also pass through this point, that is, the resistance at A acts along the beam. In the same way the resistance at A' acts along $A'B'$; and by considering the equilibrium of the system, we see that the vertical through G, the common centre of gravity, must pass through O, the point of intersection of the resistances. Now the angles which these resistances make with the normals at A and A' are equal to mOA and mOA', respectively; and the comparative safety of the equilibrium at A and A' depends on the magnitudes of these angles. Now $mOA' > mOA$. For, draw $C'q$ horizontal and Cq vertical; then, since $CG < C'G$, $qn < nC'$, and *à fortiori*, $pn < nC'$. Therefore $Am < mA'$; but $\frac{Am}{mA'} = \frac{\tan mOA}{\tan mOA'}$; therefore, $mOA' > mOA$, and if the friction were gradually diminished, the total resistance at A' would reach its limiting inclination before that at A. Hence the short beam will slip first.

4. A cylinder is supported on a rough inclined plane by a string coiled round it in a direction perpendicular to its axis, the string passing over a smooth pulley and sustaining a given mass. Find the limits to the direction of the string.

Round A, the point of contact of the cylinder and plane, describe the cone of friction, the section of which by the plane of the figure is nAm, the angles nAC and CAm being each $= \lambda$.

Let OB be any direction of the string, intersecting the vertical through the centre of the cylinder in O. Then, so long as O is between the points m and n, equilibrium is possible, because AO is a possible direction of total resistance at A. There is, of course, a particular magnitude of the weight, P, corresponding to the direction OB of the string, and this magnitude is found by taking moments about A. If θ is the angle made by the string, OB, with the inclined plane, we have

$$P = W \frac{\sin i}{2\cos^2\frac{\theta}{2}},$$

i being the inclination of the inclined plane.

If, the direction of the string being OB, P have a value greater or less than this, the cylinder will roll up or roll down the plane.

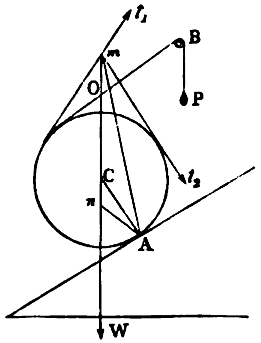

Fig. 210.

Drawing from m two tangents, mt_1 and mt_2, to the cylinder, we have the extreme directions of the string; that is, the point at which the string leaves the cylinder must lie between the points of contact of mt_1 and mt_2, on the upper portion of the cylinder; for it is evident that if the string leaves the cylinder at any point outside these limits, the point in which its line intersects that of W will be vertically above m, that is, outside the cone of friction.

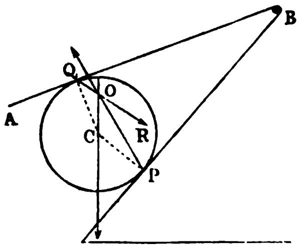

Fig. 211.

5. A heavy sphere is placed on a rough inclined plane at a point P (Fig. 211), and is kept in position by a heavy rough beam, AB, which is moveable about a fixed extremity, B, the coefficient of friction for the sphere and the beam being the same as that for the sphere and plane. Supposing that the friction is gradually diminished at both points of contact, P and Q, or that the sphere is pushed further up between the plane and beam, determine the nature of the initial motion.

The total resistances at P and Q must meet in some point, O, on the vertical through C, the centre of gravity of the sphere. Beyond this, however, their directions cannot be determined. The comparative safety of equilibrium at P and Q will depend on the relative magnitudes of the angles, CPO and CQO, which the resistances at these points make with the corresponding normals. Now it is easy to show that $CQO > CPO$; for $\sin CPO = \frac{CO}{CP} \sin COP$, and $\sin CQO = \frac{CO}{CQ} \times \sin COR$, therefore $\frac{\sin CPO}{\sin CQO} = \frac{\sin COP}{\sin COR}$; but $COR > COP$, therefore $CQO > CPO$, and if from any cause the friction is diminished, or the sphere pushed higher up, slipping must take place at Q and rolling at P.

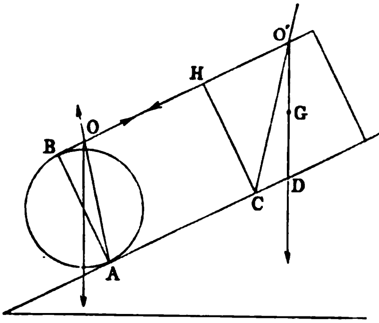

Fig. 212.

6. A cylinder is placed on a rough inclined plane, and a light rope is coiled round it in a plane perpendicular to its axis and containing its centre of gravity; this rope, after passing round the cylinder, is attached to the middle point, H (Fig. 212), of an edge of a cubical block whose height is equal to the diameter of the cylinder. Supposing the inclination of the plane to be gradually increased, determine the manner in which equilibrium will be broken, the coefficient of friction being the same for the cylinder and plane as for the cube and plane.

The motions which are here geometrically possible are—

(1) The cylinder may roll and the cube may turn over the edge C.
(2) The cylinder may roll and the cube may slip.
(3) The cylinder may slip and the cube may slip.
(4) The cylinder may slip and the cube may turn over.

Now if O is the point of intersection of the vertical through the centre of gravity of the cylinder with the rope, it is evident that the total resistance at A acts in the line OA. In the same way if O' is the point of intersection of the vertical through G, the centre of gravity of the cube, with the line of the rope, the total resistance of the plane on the cube must pass through O', and if D is the point in which the line of action of the weight of the cube intersects its base, the total resistance must evidently pass through some point between C and D.

Now this total resistance, wherever it acts, makes with the normal to the plane an angle greater than BAO; for $\tan BAO = \frac{1}{2}\tan i$, i being the inclination of the plane, and the angle which $O'D$ makes with the normal to the plane $= i$; hence the angle made with this normal by a line joining O' to any point between C and D is $> i$, and, *à fortiori*, $> BAO$. Consequently the cylinder can never slip before the cube, and cases 3 and 4 are to be rejected. The choice then is to be made between 1 and 2; and (see Art. 150) if the cube can turn over, it will do so. Hence we solve on the supposition that the cube turns over C, and if this does not require too great a value of the coefficient of friction, the cube will turn over.

The problem is to be solved by equating the values of the tension of the rope derived from the consideration of the equilibrium of the cylinder and that of the cube.

For the equilibrium of the cylinder take moments about A, and we have

$$T = \tfrac{1}{2} W \sin i, \tag{1}$$

T being the tension of the rope and W the weight of the cylinder.

Again, since by supposition the cube is about to turn round C, the total resistance of the plane acts through this point. Taking moments about C for the cube,

$$T \,.\, CH = W' \,.\, CG \sin\left(\frac{\pi}{4} - i\right),$$

or

$$T = \tfrac{1}{2} W' (\cos i - \sin i). \tag{2}$$

Equating the values of T in (1) and (2), we have

$$\tan i = \frac{W'}{W + W'}. \tag{3}$$

But in order that CO' may be a possible direction of total resistance, the angle HCO' must be $< \lambda$, or $\tan HCO' < \mu$. Now, it is easy to see that

$$\tan HCO' = \frac{1 + \tan i}{2}$$

$$= \tfrac{1}{2} \cdot \frac{W + 2W'}{W + W'}. \tag{4}$$

Hence if $\frac{1}{2}\frac{W+2W'}{W+W'} < \mu$, equilibrium will be broken by a rolling of the cylinder and turning over of the cube. If μ is less than the quantity in (4) the cylinder will roll and the cube will slip, and there is no difficulty in determining the inclination of the plane when this happens. We may either draw from O' a line making the angle of friction, λ, with the normal to the plane, and then determine T by the triangle of forces, or resolve along and perpendicular to the plane for the equilibrium of the cube. If R is the normal reaction of the plane on the cube, we find in the latter way

$$R = W' \cos i,$$

$$\mu R = W' \sin i + T;$$

therefore
$$T = W'(\mu \cos i - \sin i).$$

Equating this to the value given by (1), we have

$$\tan i = \frac{2\mu W'}{W+2W'},$$

which gives the inclination at which the cube slips.

7. Two equal carriage wheels whose centres are connected by a smooth bar are placed on a rough inclined plane; determine whether the equilibrium of the system will be best preserved by locking the hind or the fore wheel.

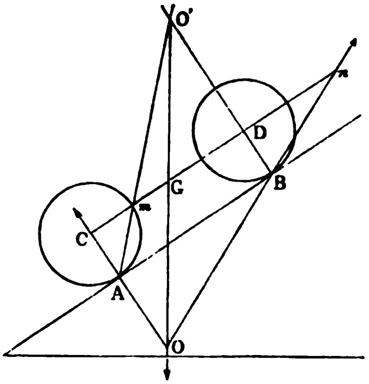

Fig. 213.

Let C and D (Fig. 213) be the centres of the wheels, and first suppose the hind wheel to be locked. Since there is no friction between the bar CD and the axle at C, the action of the bar on the lower wheel consists of a force through C (see p. 159).

The weight of this wheel also acts through C, and therefore the total resistance at A, which is the third force keeping the wheel in equilibrium, must also act through C.

Let G be the centre of gravity of the two wheels, and consider the equilibrium of the system formed by them. There are three forces acting on the system, viz., its weight through G, the total resistance at A (which has been proved to act in a line AC), and the total resistance at B. If, then, O is the point of intersection of CA and the vertical through G, the total resistance at B must act in the line OB.

We shall now determine the inclination at which equilibrium is broken.

Since the hind wheel slips, the angle $DBn = \lambda$; also let $r =$ the radius of each wheel, $CD = 2a$, and $i =$ the inclination of the plane.

Then
$$\frac{\tan COG}{\tan COn} = \frac{CG}{Cn},$$
or
$$\frac{\tan i}{\mu} = \frac{a}{2a+\mu r},$$
since $Dn = r \tan DBn = \mu r$. The inclination of the plane when equilibrium is broken is therefore given by the equation
$$\tan i = \frac{\mu a}{2a+\mu r}. \tag{1}$$

Again, suppose the fore wheel alone to be locked. In this case the total resistance at B acts in the line BD, and that at A acts in AO', O' being the intersection of BD with OG. If i' is the new inclination at which equilibrium is broken, we have, since $\angle CAO' = \lambda$,
$$\frac{\tan i'}{\mu} = \frac{DG}{Dm} = \frac{a}{2a-\mu r},$$
or
$$\tan i' = \frac{\mu a}{2a-\mu r}. \tag{2}$$

Now it is clear that i' is greater than i, and that, consequently, equilibrium will be safer when the fore wheel is locked than when the hind wheel is locked.

8. A cylinder is supported on a rough inclined plane by a light rope coiled round it in a plane perpendicular to its axis passing through its centre of gravity, the rope being attached to a fixed point. Find the direction of the rope in order that the inclination of the plane may be the greatest possible.

Let $O'B'$ (Fig. 214) be the line of the rope, and CO' the vertical through the centre of gravity of the cylinder. Then evidently the total resistance at A, the point of contact with the plane, must act in the direction AO'. If the rope took the direction OB, which is horizontal, the direction of the total resistance would be AO, and evidently the angle $CAO < CAO'$; or, in other words, the equilibrium of the cylinder will be farther from its limit when the rope is horizontal than when it takes any other direction. For a given inclination, i,

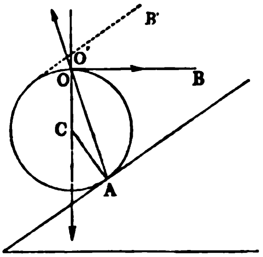

Fig. 214.

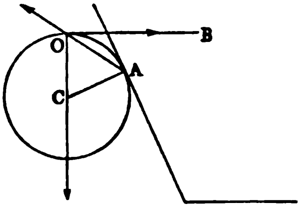

Fig. 215.

of the plane, the angle $CAO = \frac{i}{2}$, and it is clear that when CAO is equal to the angle, λ, of friction, the inclination of the plane will be at its greatest. Hence the greatest inclination of the plane $= 2\lambda$.

If the coefficient of friction be > 1, the greatest inclination of the plane will be $> \frac{\pi}{2}$, and the figure of limiting equilibrium will be that represented in Fig. 215, in which the angle CAO ($= \lambda$) is $> \frac{\pi}{4}$. But whether the cylinder will stay in this position or not depends on the initial arrangement. Unless the rope is pulled with such a force as to cause the resultant of this force and W to act in the line OA, equilibrium cannot be preserved by the resistance of the plane. In fact, unless this requisite tension of the rope is produced by pressing and scraping the cylinder against the plane, it would be possible for the cylinder to take a motion of and round its centre C which would keep its surface out of actual contact with the plane; and in this case the plane would not exert any resistance.

9. If in the preceding problem the rope, instead of being attached to a fixed point, is attached to a mass which hangs freely over a smooth pulley, find the conditions of equilibrium.

Let $O'B'$ (Fig. 214) be the direction of the rope, P the weight of the suspended mass, W the weight of the cylinder, i the inclination of the plane, λ the angle of friction, θ the angle which the rope makes with the inclined plane.

Then for equilibrium it is necessary that AO' should be the direction of total resistance at A, and that the moments of P and W about A should be equal and opposite. Hence we must have

$$CAO' = \text{or} < \lambda, \tag{1}$$

and

$$P = W \frac{\sin i}{2\cos^2 \frac{\theta}{2}}, \tag{2}$$

the second condition being equivalent to that obtained by the triangle of forces for equilibrium at O'.

If the angle $CAO' < \lambda$, and P is slightly increased above the value in (2), the initial motion will evidently be a rolling up, since moment of P about $A >$ moment of W about A; but if P is slightly diminished the rolling will be down.

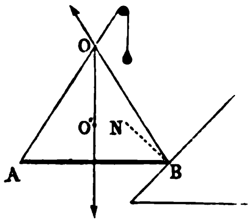

Fig. 216.

10. A heavy uniform beam, AB (Fig. 216), is to be sustained in a horizontal position, one end, B, resting on a rough inclined plane, while the other end, A, is attached to a light rope which passes over a smooth pulley and sustains a mass. Find—

(*a*) The limits to the direction of the rope, and the corresponding limiting values of the weight of the suspended mass.

(*b*) The least weight that will sustain the beam.

Let W be the weight of the beam, P the weight of the suspended mass, and BN the normal to the inclined plane at B. Then if AO be the line of the rope, intersecting the vertical through the centre of gravity of the beam in O, BO must be the direction of the total resistance at B; and in order that this may be a possible direction of total resistance, the angle NBO must be $< \lambda$, the angle of friction. Hence the limiting directions of the rope are obtained by drawing BO and BO' making the angle λ with BN on opposite sides. If the rope takes the direction AO' the beam must be on the point of slipping up, since the force of friction acts *down* the inclined plane; and if the direction of the rope is AO, the beam is on the point of slipping down. The corresponding magnitudes of P are easily determined by taking moments about B. Let p_1 and p_2 be the perpendiculars from B on AO and AO', respectively, a half the length of the beam, and P_1 and P_2 the corresponding values of P. Then

$$P_1 = W\frac{a}{p_1},$$

$$P_2 = W\frac{a}{p_2}.$$

The values of p_1 and p_2 can, of course, be easily expressed in terms of a, λ, and i, the inclination of the plane.

If the rope takes a direction intermediate to AO and AO', and if p is the length of the perpendicular from B on its direction, we have

$$P = W\frac{a}{p}.$$

Hence, if P is a minimum, p must be a maximum, since Wa is given. Now p will be a maximum when it is equal to AB, that is, when the rope is vertical. In this case the total resistance at B should also be vertical; but if the inclination of the plane $> \lambda$, this is impossible. Hence when $i > \lambda$, p is a maximum (*consistently with the conditions of the problem*) when the direction of the rope is AO; and therefore in this case P_1 is the least value of P.

If $i < \lambda$, the vertical at B is a possible direction of total resistance, and therefore AB is an admissible value of p. The corresponding value of P is therefore $\frac{1}{2}W$.

The student will easily see that if the angle of friction is greater than the complement of the inclination of the plane, there can be no limiting equilibrium in which the beam is about to slip up.

11. A cylinder is laid on a rough horizontal plane, and is in contact with a rough vertical wall; a string coiled round it at right angles to the axis passes over a smooth pulley and sustains a weight which is gradually increased till equilibrium is broken. Determine the nature of the initial motion. (Jellett's *Theory of Friction*, Example 21, p. 214.)

Let W be the weight of the cylinder, P the weight of the suspended mass, θ the angle made by the string with the horizon, λ and λ' the angles of friction at A and B, the points of contact of the cylinder with the vertical and horizontal planes, and O the point in which the line of the string intersects the vertical through C, the centre of gravity of the cylinder.

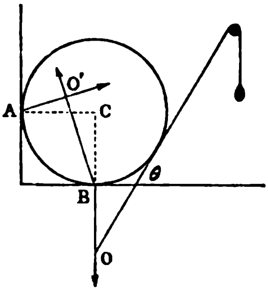

Fig. 217.

Now, in accordance with Article 150, we first consider what motions are geometrically possible. These are

(1) Rolling round A up the vertical plane.

(2) Slipping forward at B while contact ceases at A.

(3) Slipping at A and B simultaneously.

If (1) *can* happen it *will* (see Art. 150); let us suppose, therefore, that the cylinder is on the point of turning round A and coming out of contact at B. In this case there are only three forces keeping the cylinder in equilibrium, namely, W, P, and a total resistance at A. This last force should, for equilibrium, pass through O and act in the direction OA. Now whether the angle OAC is less or greater than λ, this is not a possible line of action of total resistance, because the plane cannot *pull*. Hence (1) is physically impossible.

Suppose that (2) happens. Then, as before, there are only three forces keeping the cylinder in equilibrium, namely, W, P, and the resistance at B. This last must pass through O, and must therefore act vertically. But it is obvious that such a force could not equilibrate W and P; therefore (2) is impossible.

There remains the third case, which alone is possible. To determine the value of P corresponding to limiting equilibrium, draw the lines AO' and BO' making with the normals at A and B the angles, λ and λ', of friction for the cylinder and planes. Then by taking moments about O' we easily obtain the value of P, which may also be obtained by the ordinary equations of resolution of forces. Thus, let R and R' be the normal pressures, and therefore μR and $\mu' R'$ the forces of friction, at A and B.

Taking moments about B, we have

$$R(1+\mu) = P(1-\cos\theta). \qquad (1)$$

Taking moments about A,

$$R'(1-\mu') = W - P(1+\sin\theta). \qquad (2)$$

Resolving horizontally,

$$\mu' R' - R = P\cos\theta. \qquad (3)$$

Substituting in (3) the values of R and R' given in (1) and (2), we obtain the value of P corresponding to limiting equilibrium.

It will be a useful exercise for the student to vary the position of the pulley in such a way as to render possible a case of limiting equilibrium in which the cylinder is about to ascend the vertical plane by turning round A.

12. A heavy right cone is placed with its base on a rough inclined plane, the inclination of which is gradually increased; determine whether the initial motion of the cone will be one of sliding or tumbling over.

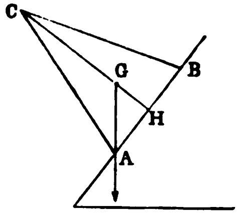

Fig. 218.

Let ABC (Fig. 218) be the vertical section of the cone through its axis, CH, and let G be the centre of gravity of the cone. (GH is $\frac{1}{4}$ CH, as will appear in a subsequent Chapter.) Then, in accordance with rule 3 of Art. 150, if it is possible for the cone to turn over the point A, the cone will do so. Solve, therefore, on the supposition that equilibrium is broken by turning round A. In this case, the two forces acting on the cone are its weight and the total resistance of the plane, which, of course, passes through A; and these forces must be equal and opposite, i.e., the total resistance must act in the vertical line AG. Now this will be possible only if AG makes with the normal to the plane an angle less than the angle of friction, λ. Hence for a tumbling motion $AGH < \lambda$. But if $\alpha = ACH$,

$$\tan AGH = 4 \tan \alpha.$$

Therefore if $\mu > 4 \tan \alpha$, the initial motion of the cone will be tumbling, and if $\mu < 4 \tan \alpha$, the initial motion will be sliding, and this sliding will evidently occur when the inclination of the plane reaches the value λ.

13. A heavy uniform bar, AB, rests on a rough horizontal table; at a given point, P, in it is attached a string which is pulled perpendicularly to AB in the plane of the table; find the point, I, about which the bar will begin to rotate when the tension of the string is sufficiently increased.

Let W = weight of bar, $AB = 2a$, $AP = c$, $BI = x$, μ = coefficient of friction. Then $\frac{W}{2a}$ is the weight per unit length = the pressure per unit length on the table, so that, when the bar is about to rotate about I, the force of friction on any element of length, ds, of the bar is $\mu \frac{W}{2a} \cdot ds$. Now these forces of friction on the portion BI will be all perpendicular to the bar in one sense, while on the portion AI they will all act in the opposite sense; also the resultant friction on BI acts at the middle point of BI and $= \mu \frac{W}{2a} \cdot BI$, or $\mu \frac{W}{2a} x$, while the force on AI acts at its middle point and $= \mu \frac{W}{2a}(2a - x)$. Again, since all the forces on the bar are just in equilibrium, the resultant of these forces of friction acts at P, opposite to the tension of the

string. Hence taking moments about P, we have

$$\mu\frac{W}{2a}x\left(2a-c-\frac{x}{2}\right) = \mu\frac{W}{2a}(2a-x)\left(a-\frac{x}{2}-c\right),$$

or $$x^2-2(2a-c)x+2a(a-c) = 0,$$

which gives two positions of I, one of which is outside the bar.

The values of x are

$$2a-c\pm\sqrt{a^2+(a-c)^2},$$

which give the following construction: at C, the middle point of AB, erect a perpendicular, CQ, equal to a, to AB; with P as centre and PQ as radius describe a circle which will cut AB in two points, I, I', one of which, I, is within AB, and the other outside it. The second is manifestly irrelevant since for the corresponding value of x the tension of the string is negative.

When $c = a$, i.e., when the string is applied at the middle point, the whole bar moves in the same sense, and the tension $= \mu W$, as is *à priori* evident.

14. A circular cylinder, or a uniform circular plate, lies on a rough horizontal table; to a given point on the circumference is attached a string which is pulled horizontally by a gradually increasing force; show that the body will begin to rotate about some point on that diameter of the base which is perpendicular to the direction of the string.

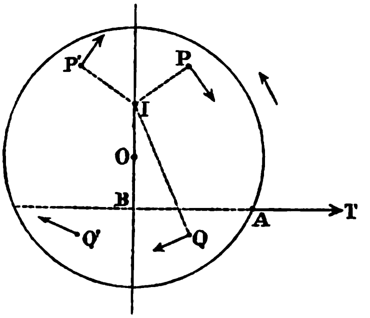

Fig. 219.

Suppose that I (Fig. 219) is the point about which rotation begins; then if W is the weight of the body and S the area of the base, the normal pressure on the base is uniformly distributed and is $\frac{W}{S}$ per unit area, so that on an element, ds, of area at any point P the force of friction will be $\mu\frac{W}{S}\cdot ds$ when rotation begins. This force is represented by the arrow at P perpendicular to PI, the rotation being supposed to take place counter-clockwise. Draw the diameter IO; then we can find a point, P', situated at the other side of this diameter, symmetrically with P, at which if we take the same element, ds, of area the force of friction will be the same as before, and will be perpendicular to $P'I$ in the sense of the arrow at P'. But these two forces give a zero component parallel to OI, their resultant being manifestly perpendicular to OI and towards the right of the figure. At all pairs of symmetrically situated points, Q, Q', below the line through I perpendicular to IO, the resultant of the forces of friction on equal elements is perpendicular to OI, but towards the left. Hence the total resultant of friction acts perpendicularly to OI;

and, if A is the point of attachment of the string and AT its direction, since the resultant of friction is exactly equal and opposite to T, the tension of the string, OI, must be perpendicular to AT.

To determine the point I, let a = radius of base, $OB = c$ = perpendicular from O on AT, and $OI = x$. Then expressing the element of surface at P with respect to I as pole, let $IP = r$, and let ϕ be the angle which IP makes with OI. The element, ds, of area at P is $rdrd\phi$, and the perpendicular from B on the line of action of the friction at P is $r + (c+x)\cos\phi$. Hence equating to zero the sum of the moments of all the forces of friction about B (which is a point on their resultant), we have

$$\int_0^\pi \int_0^R [r^2 + (c+x)\, r\cos\phi]\, d\phi dr = 0.$$

in which $R = \sqrt{a^2 - x^2\sin^2\phi} - x\cos\phi$.

Integrating first with respect to r, it is obvious that all the rational terms in x and ϕ vanish because each of them is of the form

$$\int_0^\pi \sin^{2n}\phi \,.\, \cos\phi d\phi,$$

and we have simply

$$\left(\frac{1}{3} - \frac{c}{x}\right)\int_0^\pi \Delta^3 . d\phi + \frac{c}{x}\left(1 - \frac{x^2}{a^2}\right)\int_0^\pi \Delta \,.\, d\phi = 0,$$

in which $\Delta \equiv \sqrt{1 - \frac{x^2}{a^2}\sin^2\phi}$, according to the usual notation of elliptic integrals; and of course the upper limit may be taken as $\frac{\pi}{2}$. But it is well known, by methods of reduction, that

$$\int_0^{\frac{\pi}{2}} \Delta^3 . d\phi = \frac{2}{3}\left(2 - \frac{x^2}{a^2}\right)\int_0^{\frac{\pi}{2}} \Delta \,.\, d\phi - \frac{1}{3}\left(1 - \frac{x^2}{a^2}\right)\int_0^{\frac{\pi}{2}} \frac{d\phi}{\Delta}.$$

Hence, denoting the complete integrals of the first and second kinds by K and E, respectively, the equation for x becomes

$$\left(\frac{4}{3} - \frac{c}{x} - \frac{cx}{a^2} - \frac{2x^2}{a^2}\right)E = \left(\frac{1}{3} - \frac{c}{x}\right)\left(1 - \frac{x^2}{a^2}\right)K,$$

a transcendental equation for x.

15. If one cord of a sash window breaks, find the least coefficient of friction against the sash in order that the other cord may still support the window.

Ans. $\mu = \frac{h}{b}$, where h is the height and b the breadth of the window.

16. A mass whose weight is nW is attached to a small ring which is threaded on a rough circular hoop of weight W and radius a, whose plane is vertical and which hangs from a fixed rough peg, the angle of friction between ring and hoop being λ; show that the hoop will be in equilibrium with its centre at any distance not greater than

string. Hence taking moments about P, we have

$$\mu\frac{W}{2a}x\left(2a-c-\frac{x}{2}\right)=\mu\frac{W}{2a}(2a-x)\left(a-\frac{x}{2}-c\right),$$

or $$x^2-2(2a-c)x+2a(a-c)=0,$$

which gives two positions of I, one of which is outside the bar.

The values of x are

$$2a-c\pm\sqrt{a^2+(a-c)^2},$$

which give the following construction: at C, the middle point of AB, erect a perpendicular, CQ, equal to a, to AB; with P as centre and PQ as radius describe a circle which will cut AB in two points, I, I', one of which, I, is within AB, and the other outside it. The second is manifestly irrelevant since for the corresponding value of x the tension of the string is negative.

When $c=a$, i.e., when the string is applied at the middle point, the whole bar moves in the same sense, and the tension $=\mu W$, as is *à priori* evident.

14. A circular cylinder, or a uniform circular plate, lies on a rough horizontal table; to a given point on the circumference is attached a string which is pulled horizontally by a gradually increasing force; show that the body will begin to rotate about some point on that diameter of the base which is perpendicular to the direction of the string.

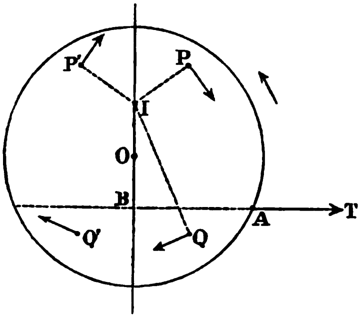

Fig. 219.

Suppose that I (Fig. 219) is the point about which rotation begins; then if W is the weight of the body and S the area of the base, the normal pressure on the base is uniformly distributed and is $\frac{W}{S}$ per unit area, so that on an element, ds, of area at any point P the force of friction will be $\mu\frac{W}{S}\cdot ds$ when rotation begins. This forc is represented by the arrow at P perpendicular to PI, the rotatio being supposed to take place counter-clockwise. Draw the diamete IO; then we can find a point, P', situated at the other side of th diameter, symmetrically with P, at which if we take the same elemen ds, of area the force of friction will be the same as before, and wi be perpendicular to $P'I$ in the sense of the arrow at P'. But the two forces give a zero component parallel to OI, their resultant bei manifestly perpendicular to OI and towards the right of the figu At all pairs of symmetrically situated points, Q, Q', below the li through I perpendicular to IO, the resultant of the forces of fricti on equal elements is perpendicular to OI, but towards the l Hence the total resultant of friction acts perpendicularly to and, if A is the point of attachment of the string and AP is [illegible] direction, since the resultant of friction [illegible] must be perpendicular [illegible] to T, the tension of the string OI, must be [illegible]

To determine the point I, let [illegible] perpendicular from O on AT, and OI [illegible] of surface at P with respect to I [illegible] the angle which IP makes with OI [illegible] circle, and the perpendicular from [illegible] friction at P [illegible] the moment [illegible] their resultant we have [illegible]

[illegible]

in which $k =$ [illegible]

Integrating [illegible]

[illegible]

and we have [illegible]

[illegible]

$\frac{n}{n+1}a\sin\lambda$ from the vertical through the peg if the coefficient of friction between the hoop and peg be not less than a certain quantity, which find. (*College Examination, Cambridge,* 1888.)

Ans. The angle of friction between hoop and peg must be $> \sin^{-1}\left(\frac{n\sin\lambda}{n+1}\right)$.

17. Two uniform bars, AB, AC, are rigidly united at A; their weights are W, W'; AB is placed on a fixed rough peg, the coefficient of friction being μ; if p is the perpendicular from C on AB, show that there is a length $\mu\frac{W'}{W+W'}\cdot p$ on AB such that if the contact with the peg is anywhere within it, the system will rest. (*Ibid.* 1884.)

18. A heavy bar passes between two rough pegs, A, B, whose distance is a, the line, AB, joining them being inclined at an angle i to the horizon, A being the higher and B the lower peg; find the limiting position of the bar.

Ans. If x is the distance of the centre of gravity of the bar from B, and μ and μ' are the coefficients of friction at A and B, respectively,

$$x = \frac{\tan i - \mu'}{\mu + \mu'}\cdot a.$$

19. A uniform heavy bar, AB, moveable in a vertical plane about a smooth horizontal axis at A, has a cord attached to B, this cord passing through a small ring fixed at a point C in the horizontal line through A such that $AC = AB$; the other end of the cord is attached to a mass P which lies on a rough horizontal table in the line AC; find the position of limiting equilibrium.

Ans. If AB makes the angle θ with AC, $\mu =$ coefficient of friction between P and the table, $W =$ mass of AB, there are two positions given by

$$2\cos^2\frac{\theta}{2} - 2\frac{\mu P}{W}\cos\frac{\theta}{2} - 1 = 0.$$

20. A uniform bar rests with its extremities on two rough inclined planes whose line of intersection is horizontal, the vertical plane through the bar being perpendicular to this line; find the limiting position of equilibrium.

Ans. If i, i' be the inclinations of the planes, λ, λ' the angles of friction between the bar and the planes, respectively, and θ the limiting inclination of the bar to the horizon,

$$2\tan\theta = \cot(i+\lambda) - \cot(i'-\lambda').$$

Another limiting position will be got by changing the sign of λ and λ'.

21. A heavy uniform rod rests with its extremities on the interior of a rough vertical circle; find the limiting position of equilibrium.

Ans. If $2a$ is the angle subtended at the centre by the rod, and λ

the angle of friction, the limiting inclination of the rod to the horizon is given by the equation

$$\tan\theta = \frac{\sin 2\lambda}{\cos 2\lambda + \cos 2\alpha}.$$

22. A heavy uniform ladder rests with one end against a rough horizontal and the other end against an equally rough vertical plane; find the least coefficient of friction that will allow the ladder to rest in all positions. *Ans.* Unity.

23. In the previous question let the centre of gravity of the ladder divide it into two segments, a and b, the latter segment being in contact with the vertical wall; given the coefficient of friction, μ, between the ladder and the ground, find the least coefficient of friction between the ladder and the wall which will allow the ladder to rest in all positions. *Ans.* $\frac{a}{\mu b}$.

24. Two equal bars, AC and CB, are connected by a smooth hinge at C, and are placed in a vertical plane with their lower extremities, A and B, resting on a rough horizontal plane; from observing the greatest value of the angle ACB for which equilibrium is possible, determine the coefficient of friction for the bars and the plane (Walton's *Mechanical Problems*, p. 96, second ed.).

Ans. If the greatest value of $\angle ACB$ is β,

$$\mu = \tfrac{1}{2}\tan\frac{\beta}{2}.$$

25. A triangular prism, whose section by a vertical plane through its centre of gravity perpendicular to its edges is ABC, rests with its base AB on a rough horizontal plane; a rope is attached to the middle point, C, of its upper edge, and, passing over a fixed pulley in the horizontal line parallel to, and in the sense of, BA, is pulled with a gradually increasing force. Find the nature of the initial motion.

Ans. If $AB = c$, $AC = b$, and the height of the prism $= h$, the prism will tilt over the edge through A if

$$\mu > \frac{c + b\cos A}{3h};$$

otherwise it will slide.

26. Prove that the work done in drawing a mass from one fixed point to another along a rough curve by a force which always acts tangentially will, with certain restrictions, be the same whatever be the shape of the curve. (*Math. Tripos*, 1884.)

27. A circular wheel is placed on a rough plane whose inclination to the horizon is gradually increased; the plane of the rim is vertical; to a given point, A, on the rim is attached one end of a rope the other end of which is attached to a fixed point, B, on the plane; find the inclination of the plane when the wheel begins to slip.

Ans. Let O be the point of contact of the rim with the plane,

C the centre of the wheel, $a = OCA$, $\beta = OBA$, $\mu =$ coefficient of friction, $i =$ inclination of plane; then

$$\tan i = \mu \frac{\cos(a+\beta) - \cos\beta}{\cos(a+\beta) + \mu\sin\beta}.$$

28. Two uniform beams are placed with their lower extremities resting on a rough horizontal plane, their upper extremities resting against each other. Show how to cut a plane face from the upper extremity of one of the beams, in order that slipping may be about to ensue at their point of contact.

Ans. Determine the line of action of their mutual resistance as in p. 229; then cut a face inclined to this line at the complement of the angle of friction.

29. A heavy uniform circular wheel rests, in a vertical plane, against the ground at A and is in contact at B with an obstacle of given height; the wheel is to be pulled over the obstacle by means of a rope (of given direction) attached at a given point to the wheel; find—

(a) The condition that the initial motion of the wheel shall be a rolling over the obstacle;

(b) The condition that the initial motion may be slipping at A and B.

(c) What ultimately happens when the initial motion is slipping at A and B.

30. Two rough inclined planes slope in the same direction and intersect in a horizontal line. A cylinder placed at their intersection and touching both all along its length has a rope coiled round it in a plane through its centre of gravity perpendicular to its axis; this rope passes over a fixed pulley and is pulled with gradually increasing force. Discuss the ways in which equilibrium may be broken by varying the tension of the rope, finding (with a given position of the rope)—

(a) The condition that must be satisfied in order that equilibrium should be possible at all;

(b) The condition that the initial motion should be one of slipping on both planes;

(c) The value of the tension of the rope when this slipping takes place.

31. A uniform bar, of which one end rests against a rough vertical wall, is supported by a light cord attached to the other end, and to a given point in the wall; find the limiting positions of equilibrium.

Ans. If $2a =$ length of bar, $2c =$ length of cord, $\lambda =$ angle of friction, $\theta =$ inclination of bar and $\phi =$ inclination of cord to the vertical, we have

$$2\cot\theta = \cot\phi - \tan\lambda$$

$$a \,.\, \sin\theta = c \,.\, \sin\phi,$$

from which both θ and ϕ may be found thus: take any two points A, B;

produce AB to C so that $AB = BC$; at C draw the line CL making $\angle ACL = \frac{\pi}{2} - \lambda$; then θ and ϕ are the angles PBC, PAC made by the lines joining B and A to a point of intersection of the line CL with the circle which is the locus of the vertex of a triangle having AB for base, the ratio of its sides AP, BP being $c : a$. Hence, in general, there are two solutions, which may, of course, become imaginary.

There is no difficulty in forming a quadratic from the above equations for $\cot\theta$, or in thence proving that there will be no limiting equilibrium unless

$$\cos\lambda > \frac{ac}{2a^2 - c^2}.$$

32. AB and CD are two equal weightless bars connected together by a smooth axis at G, their common middle point; they are placed in a vertical plane with their ends A and C resting on a rough horizontal plane, while a cord of given length connects their upper ends, B and D, and on this cord slides a ring from which a heavy body is suspended; find the limiting inclinations of the bars to the ground.

Ans. Let $AB = CD = 2a$, length of cord $= 2c$, $\lambda =$ angle of friction between the bars and the ground, $\theta =$ inclination of either bar and $\phi =$ inclination of either portion of cord to horizon. Then, if A and C are about to slip from each other,

$$2\cot\theta = \cot\phi + \tan\lambda, \qquad (1)$$

$$a\cos\theta = c\cos\phi, \qquad (2)$$

which give the following construction: draw a line $QN = a$; between N and Q take M so that $QM = c$; between M and Q take L so that $NM = ML$; at L draw the line LS making $\angle QLS = \frac{\pi}{2} - \lambda$; at Q draw QR perpendicular to QL; to this line QR draw two equal lines from N and M so that they intersect in a point, P, on the line LS; then

$$\theta = PML \text{ and } \phi = PNL.$$

If P cannot be found at the upper side of QN, the limiting equilibrium is such that A and C are about to slip towards each other, and P must be sought on the line LS at the lower side of QN; or LS may be drawn making $SLM = \frac{\pi}{2} - \lambda$.

33. A rectangular block is placed, with one of its edges horizontal, on a rough plane, the inclination of which to the horizon is gradually increased; determine whether the equilibrium of the block will be broken by a motion of sliding or one of tumbling.

Ans. If a and b are the lengths of the edges which are not horizontal, b being the length of the edge which is perpendicular to the inclined plane, the initial motion will be one of tumbling if $\mu > \frac{a}{b}$, and of sliding if $\mu < \frac{a}{b}$.

34. A cubical block is placed on a rough inclined plane and sus-

Hence if $\frac{1}{2}\,\frac{W+2W'}{W+W'} < \mu$, equilibrium will be broken by a rolling of the cylinder and turning over of the cube. If μ is less than the quantity in (4) the cylinder will roll and the cube will slip, and there is no difficulty in determining the inclination of the plane when this happens. We may either draw from O' a line making the angle of friction, λ, with the normal to the plane, and then determine T by the triangle of forces, or resolve along and perpendicular to the plane for the equilibrium of the cube. If R is the normal reaction of the plane on the cube, we find in the latter way

$$R = W' \cos i,$$

$$\mu R = W' \sin i + T;$$

therefore $$T = W'(\mu \cos i - \sin i).$$

Equating this to the value given by (1), we have

$$\tan i = \frac{2\mu W'}{W+2W'},$$

which gives the inclination at which the cube slips.

7. Two equal carriage wheels whose centres are connected by a smooth bar are placed on a rough inclined plane; determine whether the equilibrium of the system will be best preserved by locking the hind or the fore wheel.

Let C and D (Fig. 213) be the centres of the wheels, and first suppose the hind wheel to be locked. Since there is no friction between the bar CD and the axle at C, the action of the bar on the lower wheel consists of a force through C (see p. 159).

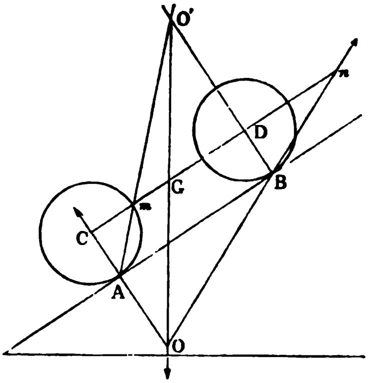

Fig. 213.

The weight of this wheel also acts through C, and therefore the total resistance at A, which is the third force keeping the wheel in equilibrium, must also act through C.

Let G be the centre of gravity of the two wheels, and consider the equilibrium of the system formed by them. There are three forces acting on the system, viz., its weight through G, the total resistance at A (which has been proved to act in a line AC), and the total resistance at B. If, then, O is the point of intersection of CA and the vertical through G, the total resistance at B must act in the line OB.

We shall now determine the inclination at which equilibrium is broken.

Since the hind wheel slips, the angle $DBn = \lambda$; also let $r =$ the radius of each wheel, $CD = 2a$, and $i =$ the inclination of the plane.

Then
$$\frac{\tan COG}{\tan COn} = \frac{CG}{Cn},$$
or
$$\frac{\tan i}{\mu} = \frac{a}{2a+\mu r},$$
since $Dn = r \tan DBn = \mu r$. The inclination of the plane when equilibrium is broken is therefore given by the equation
$$\tan i = \frac{\mu a}{2a+\mu r}. \tag{1}$$

Again, suppose the fore wheel alone to be locked. In this case the total resistance at B acts in the line BD, and that at A acts in AO', O' being the intersection of BD with OG. If i' is the new inclination at which equilibrium is broken, we have, since $\angle CAO' = \lambda$,
$$\frac{\tan i'}{\mu} = \frac{DG}{Dm} = \frac{a}{2a-\mu r},$$
or
$$\tan i' = \frac{\mu a}{2a-\mu r}. \tag{2}$$

Now it is clear that i' is greater than i, and that, consequently, equilibrium will be safer when the fore wheel is locked than when the hind wheel is locked.

8. A cylinder is supported on a rough inclined plane by a light rope coiled round it in a plane perpendicular to its axis passing through its centre of gravity, the rope being attached to a fixed point. Find the direction of the rope in order that the inclination of the plane may be the greatest possible.

Let $O'B'$ (Fig. 214) be the line of the rope, and CO' the vertical through the centre of gravity of the cylinder. Then evidently the total resistance at A, the point of contact with the plane, must act in the direction AO'. If the rope took the direction OB, which is horizontal, the direction of the total resistance would be AO, and evidently the angle $CAO < CAO'$; or, in other words, the equilibrium of the cylinder will be farther from its limit when the rope is horizontal than when it takes any other direction. For a given inclination, i,

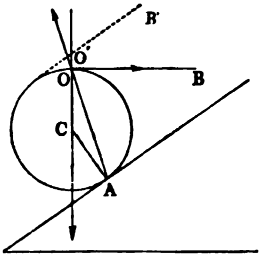

Fig. 214.

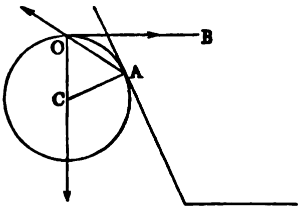

Fig. 215.

of the plane, the angle $CAO = \frac{i}{2}$, and it is clear that when CAO is equal to the angle, λ, of friction, the inclination of the plane will be at its greatest. Hence the greatest inclination of the plane $= 2\lambda$.

If the coefficient of friction be > 1, the greatest inclination of the plane will be $> \frac{\pi}{2}$, and the figure of limiting equilibrium will be that represented in Fig. 215, in which the angle CAO ($= \lambda$) is $> \frac{\pi}{4}$. But whether the cylinder will stay in this position or not depends on the initial arrangement. Unless the rope is pulled with such a force as to cause the resultant of this force and W to act in the line OA, equilibrium cannot be preserved by the resistance of the plane. In fact, unless this requisite tension of the rope is produced by pressing and scraping the cylinder against the plane, it would be possible for the cylinder to take a motion of and round its centre C which would keep its surface out of actual contact with the plane; and in this case the plane would not exert any resistance.

9. If in the preceding problem the rope, instead of being attached to a fixed point, is attached to a mass which hangs freely over a smooth pulley, find the conditions of equilibrium.

Let $O'B'$ (Fig. 214) be the direction of the rope, P the weight of the suspended mass, W the weight of the cylinder, i the inclination of the plane, λ the angle of friction, θ the angle which the rope makes with the inclined plane.

Then for equilibrium it is necessary that AO' should be the direction of total resistance at A, and that the moments of P and W about A should be equal and opposite. Hence we must have

$$CAO' = \text{or} < \lambda, \tag{1}$$

and

$$P = W \frac{\sin i}{2 \cos^2 \frac{\theta}{2}}, \tag{2}$$

the second condition being equivalent to that obtained by the triangle of forces for equilibrium at O'.

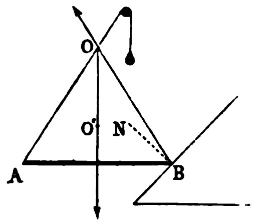

Fig. 216.

If the angle $CAO' < \lambda$, and P is slightly increased above the value in (2), the initial motion will evidently be a rolling up, since moment of P about $A >$ moment of W about A; but if P is slightly diminished the rolling will be down.

10. A heavy uniform beam, AB (Fig. 216), is to be sustained in a horizontal position, one end, B, resting on a rough inclined plane, while the other end, A, is attached to a light rope which passes over a smooth pulley and sustains a mass. Find—

(*a*) The limits to the direction of the rope, and the corresponding limiting values of the weight of the suspended mass.

(*b*) The least weight that will sustain the beam.

Let W be the weight of the beam, P the weight of the suspended mass, and BN the normal to the inclined plane at B. Then if AO be the line of the rope, intersecting the vertical through the centre of gravity of the beam in O, BO must be the direction of the total resistance at B; and in order that this may be a possible direction of total resistance, the angle NBO must be $< \lambda$, the angle of friction. Hence the limiting directions of the rope are obtained by drawing BO and BO' making the angle λ with BN on opposite sides. If the rope takes the direction AO' the beam must be on the point of slipping up, since the force of friction acts *down* the inclined plane; and if the direction of the rope is AO, the beam is on the point of slipping down. The corresponding magnitudes of P are easily determined by taking moments about B. Let p_1 and p_2 be the perpendiculars from B on AO and AO', respectively, a half the length of the beam, and P_1 and P_2 the corresponding values of P. Then

$$P_1 = W\frac{a}{p_1},$$

$$P_2 = W\frac{a}{p_2}.$$

The values of p_1 and p_2 can, of course, be easily expressed in terms of a, λ, and i, the inclination of the plane.

If the rope takes a direction intermediate to AO and AO', and if p is the length of the perpendicular from B on its direction, we have

$$P = W\frac{a}{p}.$$

Hence, if P is a minimum, p must be a maximum, since Wa is given. Now p will be a maximum when it is equal to AB, that is, when the rope is vertical. In this case the total resistance at B should also be vertical; but if the inclination of the plane $> \lambda$, this is impossible. Hence when $i > \lambda$, p is a maximum (*consistently with the conditions of the problem*) when the direction of the rope is AO; and therefore in this case P_1 is the least value of P.

If $i < \lambda$, the vertical at B is a possible direction of total resistance, and therefore AB is an admissible value of p. The corresponding value of P is therefore $\frac{1}{2}W$.

The student will easily see that if the angle of friction is greater than the complement of the inclination of the plane, there can be no limiting equilibrium in which the beam is about to slip up.

11. A cylinder is laid on a rough horizontal plane, and is in contact with a rough vertical wall; a string coiled round it at right angles to the axis passes over a smooth pulley and sustains a weight which is gradually increased till equilibrium is broken. Determine the nature of the initial motion. (Jellett's *Theory of Friction*, Example 21, p. 214.)

Let W be the weight of the cylinder, P the weight of the suspended mass, θ the angle made by the string with the horizon, λ and λ' the angles of friction at A and B, the points of contact of the cylinder with the vertical and horizontal planes, and O the point in which the line of the string intersects the vertical through C, the centre of gravity of the cylinder.

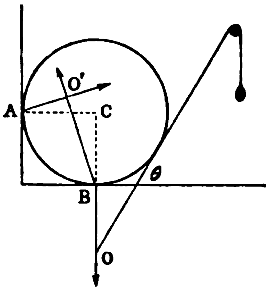

Fig. 217.

Now, in accordance with Article 150, we first consider what motions are geometrically possible. These are

(1) Rolling round A up the vertical plane.

(2) Slipping forward at B while contact ceases at A.

(3) Slipping at A and B simultaneously.

If (1) *can* happen it *will* (see Art. 150); let us suppose, therefore, that the cylinder is on the point of turning round A and coming out of contact at B. In this case there are only three forces keeping the cylinder in equilibrium, namely, W, P, and a total resistance at A. This last force should, for equilibrium, pass through O and act in the direction OA. Now whether the angle OAC is less or greater than λ, this is not a possible line of action of total resistance, because the plane cannot *pull*. Hence (1) is physically impossible.

Suppose that (2) happens. Then, as before, there are only three forces keeping the cylinder in equilibrium, namely, W, P, and the resistance at B. This last must pass through O, and must therefore act vertically. But it is obvious that such a force could not equilibrate W and P; therefore (2) is impossible.

There remains the third case, which alone is possible. To determine the value of P corresponding to limiting equilibrium, draw the lines AO' and BO' making with the normals at A and B the angles, λ and λ', of friction for the cylinder and planes. Then by taking moments about O' we easily obtain the value of P, which may also be obtained by the ordinary equations of resolution of forces. Thus, let R and R' be the normal pressures, and therefore μR and $\mu' R'$ the forces of friction, at A and B.

Taking moments about B, we have

$$R(1+\mu) = P(1-\cos\theta). \tag{1}$$

Taking moments about A,

$$R'(1-\mu') = W-P(1+\sin\theta). \tag{2}$$

Resolving horizontally,

$$\mu' R' - R = P\cos\theta. \tag{3}$$

Substituting in (3) the values of R and R' given in (1) and (2), we obtain the value of P corresponding to limiting equilibrium.

It will be a useful exercise for the student to vary the position of the pulley in such a way as to render possible a case of limiting equilibrium in which the cylinder is about to ascend the vertical plane by turning round A.

12. A heavy right cone is placed with its base on a rough inclined plane, the inclination of which is gradually increased; determine whether the initial motion of the cone will be one of sliding or tumbling over.

Let ABC (Fig. 218) be the vertical section of the cone through its axis, CH, and let G be the centre of gravity of the cone. (GH is $\frac{1}{4}$ CH, as will appear in a subsequent Chapter.) Then, in accordance with rule 3 of Art. 150, if it is possible for the cone to turn over the point A, the cone will do so. Solve, therefore, on the supposition that equilibrium is broken by turning round A. In this case, the two forces acting on the cone are its weight and the total resistance of the plane, which, of course, passes through A; and these forces must be equal and opposite, i.e., the total resistance must act in the vertical line AG. Now this will be possible only if AG makes with the normal to the plane an angle less than the angle of friction, λ. Hence for a tumbling motion $AGH < \lambda$. But if $a = ACH$,

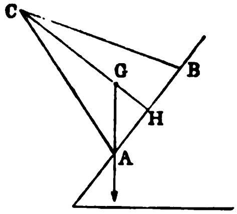

Fig. 218.

$$\tan AGH = 4 \tan a.$$

Therefore if $\mu > 4 \tan a$, the initial motion of the cone will be tumbling, and if $\mu < 4 \tan a$, the initial motion will be sliding, and this sliding will evidently occur when the inclination of the plane reaches the value λ.

13. A heavy uniform bar, AB, rests on a rough horizontal table; at a given point, P, in it is attached a string which is pulled perpendicularly to AB in the plane of the table; find the point, I, about which the bar will begin to rotate when the tension of the string is sufficiently increased.

Let W=weight of bar, $AB = 2a$, $AP = c$, $BI = x$, μ = coefficient of friction. Then $\frac{W}{2a}$ is the weight per unit length = the pressure per unit length on the table, so that, when the bar is about to rotate about I, the force of friction on any element of length, ds, of the bar is $\mu \frac{W}{2a} \cdot ds$. Now these forces of friction on the portion BI will be all perpendicular to the bar in one sense, while on the portion AI they will all act in the opposite sense; also the resultant friction on BI acts at the middle point of BI and $= \mu \frac{W}{2a} \cdot BI$, or $\mu \frac{W}{2a} x$, while the force on AI acts at its middle point and $= \mu \frac{W}{2a}(2a - x)$. Again, since all the forces on the bar are just in equilibrium, the resultant of these forces of friction acts at P, opposite to the tension of the

string. Hence taking moments about P, we have

$$\mu\frac{W}{2a}x\left(2a-c-\frac{x}{2}\right)=\mu\frac{W}{2a}(2a-x)\left(a-\frac{x}{2}-c\right),$$

or
$$x^2-2(2a-c)x+2a(a-c)=0,$$

which gives two positions of I, one of which is outside the bar.

The values of x are

$$2a-c\pm\sqrt{a^2+(a-c)^2},$$

which give the following construction: at C, the middle point of AB, erect a perpendicular, CQ, equal to a, to AB; with P as centre and PQ as radius describe a circle which will cut AB in two points, I, I', one of which, I, is within AB, and the other outside it. The second is manifestly irrelevant since for the corresponding value of x the tension of the string is negative.

When $c=a$, i.e., when the string is applied at the middle point, the whole bar moves in the same sense, and the tension $=\mu W$, as is *à priori* evident.

14. A circular cylinder, or a uniform circular plate, lies on a rough horizontal table; to a given point on the circumference is attached a string which is pulled horizontally by a gradually increasing force; show that the body will begin to rotate about some point on that diameter of the base which is perpendicular to the direction of the string.

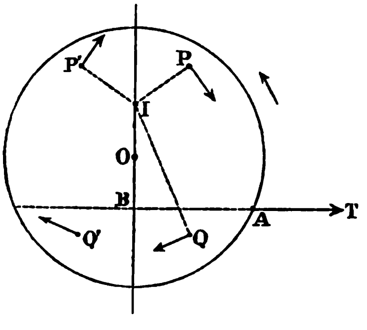

Fig. 219.

Suppose that I (Fig. 219) is the point about which rotation begins; then if W is the weight of the body and S the area of the base, the normal pressure on the base is uniformly distributed and is $\frac{W}{S}$ per unit area, so that on an element, ds, of area at any point P the force of friction will be $\mu\frac{W}{S}\cdot ds$ when rotation begins. This force is represented by the arrow at P perpendicular to PI, the rotation being supposed to take place counter-clockwise. Draw the diameter IO; then we can find a point, P', situated at the other side of this diameter, symmetrically with P, at which if we take the same element, ds, of area the force of friction will be the same as before, and will be perpendicular to $P'I$ in the sense of the arrow at P'. But these two forces give a zero component parallel to OI, their resultant being manifestly perpendicular to OI and towards the right of the figure. At all pairs of symmetrically situated points, Q, Q', below the line through I perpendicular to IO, the resultant of the forces of friction on equal elements is perpendicular to OI, but towards the left. Hence the total resultant of friction acts perpendicularly to OI;

and, if A is the point of attachment of the string and AT its direction, since the resultant of friction is exactly equal and opposite to T, the tension of the string, OI, must be perpendicular to AT.

To determine the point I, let a = radius of base, $OB = c$ = perpendicular from O on AT, and $OI = x$. Then expressing the element of surface at P with respect to I as pole, let $IP = r$, and let ϕ be the angle which IP makes with OI. The element, ds, of area at P is $rdrd\phi$, and the perpendicular from B on the line of action of the friction at P is $r + (c+x)\cos\phi$. Hence equating to zero the sum of the moments of all the forces of friction about B (which is a point on their resultant), we have

$$\int_0^\pi \int_0^R [r^2 + (c+x)\, r\cos\phi]\, d\phi dr = 0,$$

in which $R = \sqrt{a^2 - x^2\sin^2\phi} - x\cos\phi$.

Integrating first with respect to r, it is obvious that all the rational terms in x and ϕ vanish because each of them is of the form

$$\int_0^\pi \sin^{2n}\phi \,.\, \cos\phi d\phi,$$

and we have simply

$$\left(\frac{1}{3} - \frac{c}{x}\right)\int_0^\pi \Delta^3 . d\phi + \frac{c}{x}\left(1 - \frac{x^2}{a^2}\right)\int_0^\pi \Delta . d\phi = 0,$$

in which $\Delta \equiv \sqrt{1 - \frac{x^2}{a^2}\sin^2\phi}$, according to the usual notation of elliptic integrals; and of course the upper limit may be taken as $\frac{\pi}{2}$. But it is well known, by methods of reduction, that

$$\int_0^{\frac{\pi}{2}} \Delta^3 . d\phi = \frac{2}{3}\left(2 - \frac{x^2}{a^2}\right)\int_0^{\frac{\pi}{2}} \Delta . d\phi - \frac{1}{3}\left(1 - \frac{x^2}{a^2}\right)\int_0^{\frac{\pi}{2}} \frac{d\phi}{\Delta}.$$

Hence, denoting the complete integrals of the first and second kinds by K and E, respectively, the equation for x becomes

$$\left(\frac{4}{3} - \frac{c}{x} - \frac{cx}{a^2} - \frac{2x^2}{a^2}\right)E = \left(\frac{1}{3} - \frac{c}{x}\right)\left(1 - \frac{x^2}{a^2}\right)K,$$

a transcendental equation for x.

15. If one cord of a sash window breaks, find the least coefficient of friction against the sash in order that the other cord may still support the window.

Ans. $\mu = \frac{h}{b}$, where h is the height and b the breadth of the window.

16. A mass whose weight is nW is attached to a small ring which is threaded on a rough circular hoop of weight W and radius a, whose plane is vertical and which hangs from a fixed rough peg, the angle of friction between ring and hoop being λ; show that the hoop will be in equilibrium with its centre at any distance not greater than

$\frac{n}{n+1} a \sin\lambda$ from the vertical through the peg if the coefficient of friction between the hoop and peg be not less than a certain quantity, which find. (*College Examination, Cambridge*, 1888.)

Ans. The angle of friction between hoop and peg must be $> \sin^{-1}\left(\frac{n \sin\lambda}{n+1}\right)$.

17. Two uniform bars, AB, AC, are rigidly united at A; their weights are W, W'; AB is placed on a fixed rough peg, the coefficient of friction being μ; if p is the perpendicular from C on AB, show that there is a length $\mu \frac{W'}{W+W'} \cdot p$ on AB such that if the contact with the peg is anywhere within it, the system will rest. (*Ibid.* 1884.)

18. A heavy bar passes between two rough pegs, A, B, whose distance is a, the line, AB, joining them being inclined at an angle i to the horizon, A being the higher and B the lower peg; find the limiting position of the bar.

Ans. If x is the distance of the centre of gravity of the bar from B, and μ and μ' are the coefficients of friction at A and B, respectively,

$$x = \frac{\tan i - \mu'}{\mu + \mu'} \cdot a.$$

19. A uniform heavy bar, AB, moveable in a vertical plane about a smooth horizontal axis at A, has a cord attached to B, this cord passing through a small ring fixed at a point C in the horizontal line through A such that $AC = AB$; the other end of the cord is attached to a mass P which lies on a rough horizontal table in the line AC; find the position of limiting equilibrium.

Ans. If AB makes the angle θ with AC, $\mu =$ coefficient of friction between P and the table, $W =$ mass of AB, there are two positions given by

$$2\cos^2\frac{\theta}{2} - 2\frac{\mu P}{W}\cos\frac{\theta}{2} - 1 = 0.$$

20. A uniform bar rests with its extremities on two rough inclined planes whose line of intersection is horizontal, the vertical plane through the bar being perpendicular to this line; find the limiting position of equilibrium.

Ans. If i, i' be the inclinations of the planes, λ, λ' the angles of friction between the bar and the planes, respectively, and θ the limiting inclination of the bar to the horizon,

$$2\tan\theta = \cot(i+\lambda) - \cot(i'-\lambda').$$

Another limiting position will be got by changing the sign of λ and λ'.

21. A heavy uniform rod rests with its extremities on the interior of a rough vertical circle; find the limiting position of equilibrium.

Ans. If $2a$ is the angle subtended at the centre by the rod, and λ

the angle of friction, the limiting inclination of the rod to the horizon is given by the equation

$$\tan\theta = \frac{\sin 2\lambda}{\cos 2\lambda + \cos 2a}.$$

22. A heavy uniform ladder rests with one end against a rough horizontal and the other end against an equally rough vertical plane; find the least coefficient of friction that will allow the ladder to rest in all positions. *Ans.* Unity.

23. In the previous question let the centre of gravity of the ladder divide it into two segments, a and b, the latter segment being in contact with the vertical wall; given the coefficient of friction, μ, between the ladder and the ground, find the least coefficient of friction between the ladder and the wall which will allow the ladder to rest in all positions. *Ans.* $\frac{a}{\mu b}$.

24. Two equal bars, AC and CB, are connected by a smooth hinge at C, and are placed in a vertical plane with their lower extremities, A and B, resting on a rough horizontal plane; from observing the greatest value of the angle ACB for which equilibrium is possible, determine the coefficient of friction for the bars and the plane (Walton's *Mechanical Problems*, p. 96, second ed.).

Ans. If the greatest value of $\angle ACB$ is β,

$$\mu = \tfrac{1}{2}\tan\frac{\beta}{2}.$$

25. A triangular prism, whose section by a vertical plane through its centre of gravity perpendicular to its edges is ABC, rests with its base AB on a rough horizontal plane; a rope is attached to the middle point, C, of its upper edge, and, passing over a fixed pulley in the horizontal line parallel to, and in the sense of, BA, is pulled with a gradually increasing force. Find the nature of the initial motion.

Ans. If $AB = c$, $AC = b$, and the height of the prism $= h$, the prism will tilt over the edge through A if

$$\mu > \frac{c + b\cos A}{3h};$$

otherwise it will slide.

26. Prove that the work done in drawing a mass from one fixed point to another along a rough curve by a force which always acts tangentially will, with certain restrictions, be the same whatever be the shape of the curve. (*Math. Tripos*, 1884.)

27. A circular wheel is placed on a rough plane whose inclination to the horizon is gradually increased; the plane of the rim is vertical; to a given point, A, on the rim is attached one end of a rope the other end of which is attached to a fixed point, B, on the plane; find the inclination of the plane when the wheel begins to slip.

Ans. Let O be the point of contact of the rim with the plane,

C the centre of the wheel, $a = OCA$, $\beta = OBA$, $\mu =$ coefficient of friction, $i =$ inclination of plane; then

$$\tan i = \mu \frac{\cos(a+\beta) - \cos\beta}{\cos(a+\beta) + \mu \sin\beta}.$$

28. Two uniform beams are placed with their lower extremities resting on a rough horizontal plane, their upper extremities resting against each other. Show how to cut a plane face from the upper extremity of one of the beams, in order that slipping may be about to ensue at their point of contact.

Ans. Determine the line of action of their mutual resistance as in p. 229; then cut a face inclined to this line at the complement of the angle of friction.

29. A heavy uniform circular wheel rests, in a vertical plane, against the ground at A and is in contact at B with an obstacle of given height; the wheel is to be pulled over the obstacle by means of a rope (of given direction) attached at a given point to the wheel; find—

(*a*) The condition that the initial motion of the wheel shall be a rolling over the obstacle;

(*b*) The condition that the initial motion may be slipping at A and B.

(*c*) What ultimately happens when the initial motion is slipping at A and B.

30. Two rough inclined planes slope in the same direction and intersect in a horizontal line. A cylinder placed at their intersection and touching both all along its length has a rope coiled round it in a plane through its centre of gravity perpendicular to its axis; this rope passes over a fixed pulley and is pulled with gradually increasing force. Discuss the ways in which equilibrium may be broken by varying the tension of the rope, finding (with a given position of the rope)—

(*a*) The condition that must be satisfied in order that equilibrium should be possible at all;

(*b*) The condition that the initial motion should be one of slipping on both planes;

(*c*) The value of the tension of the rope when this slipping takes place.

31. A uniform bar, of which one end rests against a rough vertical wall, is supported by a light cord attached to the other end, and to a given point in the wall; find the limiting positions of equilibrium.

Ans. If $2a =$ length of bar, $2c =$ length of cord, $\lambda =$ angle of friction, $\theta =$ inclination of bar and $\phi =$ inclination of cord to the vertical, we have

$$2\cot\theta = \cot\phi - \tan\lambda$$

$$a \, . \sin\theta = c \, . \sin\phi,$$

from which both θ and ϕ may be found thus: take any two points A, B;

produce AB to C so that $AB = BC$; at C draw the line CL making $\angle ACL = \frac{\pi}{2} - \lambda$; then θ and ϕ are the angles PBC, PAC made by the lines joining B and A to a point of intersection of the line CL with the circle which is the locus of the vertex of a triangle having AB for base, the ratio of its sides AP, BP being $c : a$. Hence, in general, there are two solutions, which may, of course, become imaginary.

There is no difficulty in forming a quadratic from the above equations for $\cot\theta$, or in thence proving that there will be no limiting equilibrium unless

$$\cos\lambda > \frac{ac}{2a^2 - c^2}.$$

32. AB and CD are two equal weightless bars connected together by a smooth axis at G, their common middle point; they are placed in a vertical plane with their ends A and C resting on a rough horizontal plane, while a cord of given length connects their upper ends, B and D, and on this cord slides a ring from which a heavy body is suspended; find the limiting inclinations of the bars to the ground.

Ans. Let $AB = CD = 2a$, length of cord $= 2c$, $\lambda =$ angle of friction between the bars and the ground, $\theta =$ inclination of either bar and $\phi =$ inclination of either portion of cord to horizon. Then, if A and C are about to slip from each other,

$$2\cot\theta = \cot\phi + \tan\lambda, \tag{1}$$

$$a\cos\theta = c\cos\phi, \tag{2}$$

which give the following construction: draw a line $QN = a$; between N and Q take M so that $QM = c$; between M and Q take L so that $NM = ML$; at L draw the line LS making $\angle QLS = \frac{\pi}{2} - \lambda$; at Q draw QR perpendicular to QL; to this line QR draw two equal lines from N and M so that they intersect in a point, P, on the line LS; then

$$\theta = PML \text{ and } \phi = PNL.$$

If P cannot be found at the upper side of QN, the limiting equilibrium is such that A and C are about to slip towards each other, and P must be sought on the line LS at the lower side of QN; or LS may be drawn making $SLM = \frac{\pi}{2} - \lambda$.

33. A rectangular block is placed, with one of its edges horizontal, on a rough plane, the inclination of which to the horizon is gradually increased; determine whether the equilibrium of the block will be broken by a motion of sliding or one of tumbling.

Ans. If a and b are the lengths of the edges which are not horizontal, b being the length of the edge which is perpendicular to the inclined plane, the initial motion will be one of tumbling if $\mu > \frac{a}{b}$, and of sliding if $\mu < \frac{a}{b}$.

34. A cubical block is placed on a rough inclined plane and sus-

tained by a rope, parallel to the inclined plane, attached to the middle point of the upper edge (which is horizontal); the rope lies in the vertical plane which contains the centre of the cube and is perpendicular to the inclined plane. Shew that the greatest inclination of the plane is

$$\frac{\pi}{4}+\tan^{-1}\left(\frac{\mu}{1+\mu}\right).$$

35. A solid triangular prism is placed, with its axis horizontal, on a rough inclined plane, the inclination of which is gradually increased; determine the nature of the initial motion of the prism.

Ans. If the triangle ABC, of area Δ, is the section perpendicular to the axis, and the side AB is in contact with the plane, A being the lower vertex, the initial motion will be one of tumbling if

$$\mu > \frac{b^2+3c^2-a^2}{4\Delta}.$$

36. A frustum of a solid right cone is placed with its base on a rough inclined plane, the inclination of which is gradually increased; determine the nature of the initial motion of the body.

Ans. If the radii of the larger and smaller sections are R and r, and h is the height of the frustum, the initial motion will be one of tumbling or slipping according as

$$\mu >< \frac{4R}{h}\cdot\frac{R^2+Rr+r^2}{R^2+2Rr+3r^2}.$$

37. $ABCD$ is a section of a heavy rectangular block through its centre of gravity parallel to one of its faces; it is placed with AB on a rough horizontal plane; a bar rests against the vertical face AD at a given point E, and against the ground at a point F in BA produced; a gradually increasing load is suspended from a point G in the bar; determine how and when equilibrium will be broken, with the following numerical data:—

$AB = 6$, $AE = 8$, $FG = 7$, $GE = 10$, μ along $AB = \frac{1}{2}$,

μ' at $E = \frac{1}{4}$, and μ'' at $F =$ (firstly) $\frac{1}{2}$, and (secondly) $\frac{3}{4}$.

Ans. With the first value of μ'' (viz., $\frac{1}{2}$), the bar will begin to slide before any load at all is applied, the block remaining at rest. With the second value of μ'', when the load has reached the value $\frac{789}{910}W$, the block will begin to turn over B, and the bar to revolve about F while slipping between them takes place at E, W being the weight of the block.

38. A ladder, AB, 15 feet long, rests against the ground at A and against an equally rough vertical wall at B; to a point D in the ladder at a distance of 10 feet from A is attached a rope which, passing over a pulley at a point on the production through D of the line joining D to the intersection of the vertical plane and the ground, sustains a weight equal to half that of the ladder; the centre of gravity, G, of the ladder is 6 feet from A; the coefficients of friction are each equal to $\frac{1}{2}$; find the limiting inclination of the ladder.

Ans. $\tan^{-1}\frac{2}{3}$.

39. An elliptic cylinder is placed, with its axis horizontal, on a rough plane inclined to the horizon at an angle less than the angle of friction; prove that the cylinder cannot rest if the eccentricity of the section perpendicular to the axis is less than $\sqrt{\dfrac{2\sin i}{1+\sin i}}$, i being the inclination of the plane.

40. A cylinder, the section of which perpendicular to the axis is any given curve, is to be placed, with the axis horizontal, on a rough inclined plane; how must it be placed so that it shall be least likely to slip, the cylinder being in contact with the plane along a single line?

41. An elliptic cylinder rests in limiting equilibrium between a rough vertical and an equally rough horizontal plane, the axis of the cylinder being horizontal, and the major axis of the ellipse inclined to the horizon at an angle of 45°. Find the coefficient of friction.

Ans.
$$\mu = \frac{\sqrt{1+2e^2-e^4}-1}{2-e^2},$$

e being the eccentricity of the ellipse. (Employ the Theorem of Art. 36.)

42. The circumstances of the preceding problem remaining the same, except that the vertical plane is smooth, show that the coefficient of friction is $\frac{1}{4}e^2$ (Walton's *Mechanical Problems*, p. 82).

If the horizontal plane alone is smooth, is it possible for the cylinder to rest in any position?

43. A uniform circular plate lying with its face on a rough inclined plane can turn in its plane about a point on its rim, find the condition that it should rest in all positions. (*Cambridge Examinations*, 1888.)

Ans. If μ is the coefficient of friction, i = inclination of plane, the condition is
$$\mu > \frac{9\pi}{32}\tan i.$$

If this is not satisfied, the greatest angle, θ, that the line joining the fixed point to the centre can make with the lines of greatest slope is given by the equation
$$\sin\theta = \frac{32\mu}{9\pi}\cot i.$$

44. A cylinder is placed on a rough horizontal plane, and a uniform plank rests with one end on the ground and the other against the cylinder (the plank being at right angles to the axis of the cylinder). If the plank is gradually lowered until equilibrium is about to be broken, show that slipping will take place only at the point of contact of the plank and cylinder, whatever be their dimensions. For any position of the plank find the direction of the reaction of the ground on the cylinder.

Ans. If θ is the angle made by the plank with the ground, P = weight of plank, W = weight of cylinder, r = radius of cylinder,

$2a$ = length of plank, ψ = angle made with the vertical by the reaction of the ground on the cylinder,

$$\cot\psi = \frac{2W}{P}\tan\theta + \left(1+\frac{2W}{P}\right)\sqrt{\frac{a\sin\theta}{r-a\sin\theta}}.$$

45. A cylinder placed on a rough plane has a string coiled round it in a plane at right angles to its axis; the string after passing round the cylinder is attached to a heavy particle which also rests on the plane. If the plane is gradually tilted up, determine the nature of the initial motion.

Ans. The cylinder will roll and the particle slip if both are equally rough; and if i is the inclination of the plane when this happens,

$$\tan i = \frac{2\mu P\cos^2 a}{W\cos 2a + 2P\cos^2 a + \mu W\sin 2a},$$

where W and P are the weights of the cylinder and the particle, μ the coefficient of friction, and $2a$ the angle between the string and the inclined plane.

46. A heavy cylinder is laid on a rough inclined plane, its axis being horizontal; a heavy uniform plank rests on the cylinder and against the inclined plane, the plank being horizontal at right angles to the axis of the cylinder, and touching the cylinder at its highest point. Supposing the inclination of the plane to be gradually increased, the horizontality of the plank being always preserved, determine the nature of the initial motion of the system and the inclination of the plane at which equilibrium is broken.

Ans. The plank will slip at its point of contact with the plane, a rolling motion taking place at the other points of contact in the system; and the inclination (i) is given by the equation

$$\left(\frac{r}{a}\cot\frac{i}{2}-1\right)\left[P\cot\frac{i}{2}\tan(\lambda-i)-W\right] = P+W,$$

where r = radius of cylinder, $2a$ = length of plank, W = weight of cylinder, P = weight of plank, and λ = angle of friction.

47. Two particles, A and B, whose weights are denoted by A and B, are connected by a string fully stretched, and placed on a rough horizontal plane, the coefficient of friction for each particle being μ. A force P, which is $<\mu(A+B)$, is applied to A in the direction BA, and its direction is gradually turned round through an angle θ in the plane. Find the nature of the initial motion of the system.

Ans. If $P<\mu\sqrt{A^2+B^2}$ and $>\mu A$, the particle A alone will slip, and this happens when $\sin\theta = \frac{\mu A}{P}$. If $P>\mu\sqrt{A^2+B^2}$, both will slip when $\cos\theta = \frac{P^2+\mu^2(B^2-A^2)}{2\mu BP}$.

48. A heavy rod is placed in any manner resting on two points A and B of a rough horizontal curve, and a string attached to a point C of the chord AB is pulled in any direction in the plane of the curve so

that the rod is on the point of motion. Prove that the locus of the intersection of the lines of action of the frictions at A and B is an arc of a circle and a part of a straight line; except when C is the centre of gravity of the rod, in which case the directions of the frictions will be always parallel to the string.

49. A bar rests with two short legs, fixed at A and B, on a rough horizontal plane; a cord is attached to the bar at a given point, M, and pulled horizontally with gradually increasing force; find the position of the point, I, about which the bar will begin to turn.

Ans. Let P, Q be the pressures on the legs A, B; describe the circular locus of the vertex, C, of a triangle whose base is AB and ratio of sides $CA : CB = P \,.\, AM : Q \,.\, BM$; if this circle intersects the cord in C, the point I is the intersection of perpendiculars at A and B to CA and CB, respectively.

50. A uniform bar of weight W, bent into a semicircle, rests on a rough horizontal table; AB is the diameter joining its ends and C is the middle point of the arc. A string tied to C is pulled gently in the direction CA, and the tension is increased until the bar begins to move; show that the tension at this instant is equal to

$$\frac{2\sqrt{2}\mu W}{\pi}.$$

(*St. John's College*, 1886.)

51. A uniform circular ring (or a hollow circular cylinder) of small thickness is placed with its axis vertical on a rough horizontal plane; if a string attached to a given point in the body is pulled with gradually increasing force in a given horizontal line, find the point about which the body will begin to rotate.

Ans. If a = radius of the ring, c = perpendicular from centre on the string, the point of turning will be on the diameter perpendicular to the string at a distance x from the centre given by the equation

$$(c-x)(a+x) \,.\, E = (c+x)(a-x) \,.\, K,$$

where E and K are the complete elliptic integrals of the second and first kinds with modulus $\dfrac{2\sqrt{ax}}{a+x}$.

CHAPTER XI.

CENTROIDS AND CENTRES OF MASS.

159.] **Theorem of Mass Moments.** We have already (see Art. 90) defined the centre of mass of any system of particles, whether they form a continuous body or not, and shown how to find the distance of this point from any plane.

In the case of a continuous body, if we take any point, P, in it, and at P take an indefinitely small element, dm, of mass, and multiply dm by the distance, z, of P from the plane of reference, we obtain the mass moment,

$$z\,dm,$$

of the particle with respect to the plane. The sum of these mass moments for all the elements of mass of the body is equal to the product of the whole mass of the body, M, and the distance, $\bar{z}$, of the centre of mass of the body from the plane; so that

$$\bar{z} = \frac{\int z\,dm}{M}. \tag{1}$$

This gives the distance of the centre of mass, G, from the particular plane of reference; but to determine the point completely, we require, in general, to know its distances from any two other planes of reference.

If through any fixed point, O, there be drawn any three lines Ox, Oy, Oz, usually taken rectangular, and if x, y, z denote the co-ordinates of P, the position of the indefinitely small element, dm, of mass, and if $\bar{x}$, $\bar{y}$, $\bar{z}$ denote the co-ordinates of G, we have in addition to (1) the equations

$$\bar{x} = \frac{\int x\,dm}{M}; \quad \bar{y} = \frac{\int y\,dm}{M}. \tag{2}$$

These expressions hold whether the co-ordinates are rectangular or oblique.

Sometimes, owing to some simplicity of the figure of the given body, we know at the outset a particular line on which G lies, and in such cases the point G will become known if we calculate its distance from any *one* plane.

We may obviously define the centre of mass of the body as the point whose distance from any plane is the mean distance of the given mass from the plane.

Let it be particularly observed that no such theorem of mass moments holds with respect to a *line*. Given any right line, Ox, we may, indeed, calculate the mean distance of the body from the line thus: if p is the perpendicular distance of the point P of the body from Ox, the expression

$$\frac{\int p\,dm}{M}$$

gives the mean distance of the body from the line, and all points lying on a cylinder of this radius having Ox for its axis are at the mean distance of the body from the axis; but the *centre of mass* of the body, defined as in p. 129—i.e., the centre of mean position of all its points for multiples proportional to the elements of mass at these points—is not, in general, one of these points; and, moreover, there is no single point in the body such that its distance from *every* line in space is the mean distance of the body itself from that line, whereas the unique property *does* belong to the centre of mass—i.e., the distance of this point from *every plane* is the mean distance of the body from the plane, as is at once seen from the method by which this point, G, is deduced (Art. 90), a method which consists in dividing the lines joining particle to particle in ratios which have nothing to do with any planes of reference. This uniqueness of the point G may also be verified analytically from the expressions (1), (2) for its distances from three particular planes of reference. For, let any other plane of reference be taken, and let the equation of this plane with respect to the three selected axes Ox, Oy, Oz be

$$\lambda x + \mu y + \nu z - p = 0, \tag{3}$$

where λ, μ, ν are the three direction-cosines of its perpendicular from O and p is the length of this perpendicular. Then if x, y, z are the co-ordinates of the point P in the body, the length of the perpendicular from P on the plane (3) is

$$\lambda x + \mu y + \nu z - p,$$

so that the mean distance of the body from the plane is

$$\frac{1}{M}\int(\lambda x+\mu y+\nu z-p)\,dm,$$

or
$$\frac{1}{M}(\lambda\int x\,dm+\mu\int y\,dm+\nu\int z\,dm-Mp),$$

or
$$\lambda\bar{x}+\mu\bar{y}+\nu\bar{z}-p,$$

which is precisely the distance of the point G from the plane. Hence if G is determined with reference to *any* three planes, we shall arrive at the very same point whatever other planes of reference we take for its determination.

It has been already pointed out why the centre of mass of a body or any system of bodies coincides with its *centre of gravity*, when we consider the system as acted upon by the attraction of the earth.

We might, in the very same way, define the *centre of volume* of a body. Thus, if P is any point in the body, and if dv is an indefinitely small element of volume described round P, V being the volume of the whole body, the expression

$$\frac{\int z\,dv}{V}$$

is the distance of the mean volume-point of the body from a plane, and this will coincide with the distance of the centre of mass if the body is homogeneous; for, if ρ is the mass per unit volume at P, the mass inside dv is ρdv, so that the distance of the centre of mass is
$$\frac{\int \rho z\,dv}{M},$$

which agrees with the previous expression if ρ is the same at all points, since it can then be taken outside the sign of integration.

If instead of a material body we have a *surface* of any form—a mere area—we can define in the same way the point whose distance from any plane is the mean distance of the surface from the plane. Thus, if P is any point on the surface and if dS is an indefinitely small element of the area described round P, the whole area of the surface being S, the expression

$$\frac{\int z\,dS}{S} \qquad (4)$$

is the mean distance of the surface from the plane from which z is reckoned; and replacing dm by dS in (1) and (2), and

M by S, we arrive at the co-ordinates of the point G which is sometimes spoken of as the centre of gravity of the surface. This point is now more usually called the *centroid* of the surface, or its *centre of area*. If instead of a mere geometrical surface we imagine S to be an indefinitely thin membrane of uniform thickness and density, the centroid may be regarded as the centre of mass of the membrane.

Given two bodies of any shapes and sizes and the centre of mass of each, to find the centre of mass of the two combined it is not necessary to resort to any integration. We have merely to join their centres of mass, g_1 and g_2, and on the joining line to take a point, G, dividing it into two segments inversely proportional to their masses, m_1 and m_2, respectively; thus

$$\frac{Gg_1}{Gg_2} = \frac{m_2}{m_1}.$$

And if z_1, z_2 are the distances of g_1, g_2 from any plane, and $\bar{z}$ the distance of G from the plane,

$$(m_1 + m_2)\,\bar{z} = m_1 z_1 + m_2 z_2.$$

Again, given the centre of mass, G, of a mass M, and also the centre of mass, g_1, of a portion, m_1, of the mass, the centre of mass, g_2, of the remainder is a point on the line g_1G produced through G, such that $\frac{Gg_2}{Gg_1} = \frac{m_1}{M - m_1}$.

An obvious and important property of the centre of mass of any material system is that *the mass moment of the system with respect to any plane passing through its centre of mass is zero*; that is, if $m_1, m_2, \ldots$ are any masses whose centres of mass are distant $z_1, z_2, \ldots$ from any plane passing through G,

$$m_1 z_1 + m_2 z_2 + m_3 z_3 + \ldots = 0; \qquad (5)$$

and conversely, if the mass moment with respect to a plane vanishes, the plane passes through the centre of mass.

If the system forms a continuous body,

$$\int z\,dm = 0, \qquad (6)$$

for any such plane; and for a continuous surface

$$\int z\,dS = 0. \qquad (7)$$

The possibility of the vanishing of the mass moment depends, of course, on the fact that the distances of points at opposite sides of the plane of reference from this plane are affected with opposite signs, i.e., some of the z's in (5) are positive while others are negative.

Hence a fundamental difference between the mass moment

$$\int z\,dm \tag{8}$$

with reference to a plane and another kind of mass moment,

$$\int z^2 dm, \tag{9}$$

which has often in Kinetics to be calculated for the system, i.e., an expression in which each element of mass is multiplied by the *square* of its distance from the plane. This latter expression can never vanish for any plane—unless we imagine a system in which some elements of mass are positive and some negative—since, whether z is $+$ or $-$ in (9), each term of the integral is $+$. The expression (9) when divided by M, the whole mass of the system, gives the *mean square of distance* of the body from the plane, and this must be carefully distinguished from the *square of the mean distance* of the body from the plane. The latter is $\bar{z}^2$, the square of the distance of G from the plane, and it can vanish, of course. It is well, in view of errors which are not unfrequently made, to call the attention of the student to the fact that though for many purposes in Statics and Kinetics, a material system may be considered as concentrated at its centre of mass, it cannot for all purposes be so considered. We have just seen that for the calculation of the mean square of distance of the system from a given plane it cannot be concentrated at G; and there are many other cases—as, for instance, in the calculation of Attraction and of Moment of Inertia about an axis—in which it is not allowable to concentrate a body at its centre of mass.

160.] **Measure of Density.** When a body is of the same constitution throughout, i.e., when its ultimate particles are undistinguishable from each other, and when there is the same number of them in a given volume wherever this volume is taken in the body, the body is said to be *homogeneous* or of *uniform density*; and its density is measured by the quantity of matter contained in (some selected) unit of volume. But when the particles are more or less crowded together in one region of the body than in another, instead of speaking of the density of the body, we must speak of the density *at each particular point*. To measure this, take any very small volume, dv, round the point, and let dm be the quantity of matter contained in it;

then the limiting value of the ratio $\frac{dm}{dv}$, when dv (and therefore dm) is indefinitely diminished, is the *density of the body at the point considered.* Density may be measured in grammes per cubic centimètre, or in pounds per cubic foot.

We now proceed to the more elementary cases of the determination of Centres of Mass.

SECTION I.

Elementary Cases.

161.] **Centre of Mass of a Triangular Lamina of Uniform Thickness and Density.** Let ABC be any triangular lamina of uniform thickness and density, and let it be divided by an indefinitely great number of lines parallel to the base BC into an indefinitely great number of strips. Then the centre of mass of each strip is its middle point; and the middle points of all the strips lie on the line joining A to the middle point of BC. Hence the centre of mass of the lamina lies on this line. Similarly, the centre of mass lies on the line joining B to the middle point of CA. It is therefore *the intersection of the bisectors of the sides drawn from the opposite vertices.*

Again, *the centre of mass of a uniform triangular lamina coincides with the centre of mass of three equal particles placed at its vertices.*

For, the centre of mass of the two equal particles at B and C is the middle point of BC, and the centre of mass of the three lies on the line joining this point to A. Similarly, it lies on the line joining B to the middle point of CA. Therefore, &c.

If the mass of each particle is m, the centre of mass divides the line joining A to the middle of BC in the ratio $2m:m$, or $2:1$. Hence *the centre of mass of a triangular lamina of uniform thickness and density lies on the bisector of any side drawn from the opposite vertex at the point of trisection (nearest to the side) of the bisector.*

COR. If the distances (rectangular or oblique) of the vertices of a triangle from *any plane* are x_1, x_2, and x_3, the distance of its centre of mass from this plane is $\frac{x_1+x_2+x_3}{3}$.

162.] **Centre of Mass of a Triangular Pyramid of Uniform Density.** Let $ABCD$ (Fig. 220) be a triangular pyramid. Now if any vertex, D, be joined to the centroid, N, of the opposite face, the joining line passes through the centroids of all triangles in which the pyramid is cut by planes parallel to this face. For, let abc be a section of the pyramid parallel to the base, ABC. Draw the plane CND containing the lines CD and DN; this plane bisects the base AB in H, since (Art. 161) CN bisects AB. Let the plane CND intersect the face ABD in the right line HhD, h being the point in which this line meets ab. Then since in the triangle ABD, ab is parallel to AB, and DH bisects AB, h is the middle point of ab.

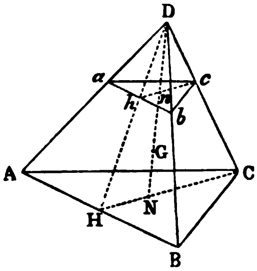

Fig. 220.

Again, if the line DN meets the plane abc in n, the points h, n, and c are in a right line. For these are evidently points common to the planes CND and abc, and since two planes intersect in a right line, the points h, n, c are in a right line—that is to say, n is a point on the bisector of the side ab drawn through c. Similarly, n is a point on the bisector of bc drawn through a; therefore n is the centroid of the triangle abc.

To find the centre of mass of the pyramid, let it be divided by planes parallel to ABC into an indefinitely great number of triangular laminæ. Now we have just proved that the centres of mass of all these laminæ lie on the line, DN, joining the vertex D to the centroid of the opposite base. Similarly, the centre of mass of the pyramid lies on the line joining the vertex A to the centroid of the face BCD. It is, therefore, the point, G, of intersection of lines drawn from any two vertices to the centroids of the opposite faces. But this is exactly the construction for the centre of mass of a system of four equal particles placed at the vertices of the pyramid. Hence—

The centre of mass of a triangular pyramid coincides with the centre of mass of four equal particles placed at its vertices.

Also—

The centre of mass of a triangular pyramid is one-fourth of the way up the line joining the centroid of any face to the opposite vertex.

For, if at the vertices there be placed four equal particles, each of mass m, their centre of mass is found by joining D to N and taking $\frac{GN}{GD} = \frac{m}{3m} = \frac{1}{3}$, therefore $GN = \frac{1}{3}GD$, or

$$NG = \tfrac{1}{4}ND.$$

Cor. 1. The *perpendicular* distance of the centre of mass of a triangular pyramid from the base is equal to $\frac{1}{4}$ height of pyramid.

Cor. 2. If the distances (rectangular or oblique) of the vertices of a pyramid from *any plane* are x_1, x_2, x_3, x_4, the distance of the centre of mass from the plane is $\frac{x_1 + x_2 + x_3 + x_4}{4}$.

163.] **Centre of Mass of a Cone of Uniform Density having any Plane Base.** Consider a pyramid whose base is a polygon of any number of sides. Then, by dividing the base into triangles we can consider the whole pyramid as composed of a number of triangular pyramids. Now (Art. 162) the centre of mass of each of these pyramids lies in a plane whose distance from the base is one-fourth of the height of the pyramid; therefore the centre of mass of the whole pyramid lies in this plane—that is, its *perpendicular* distance from the base is one-fourth of the height of the pyramid.

Again, dividing the pyramid into an indefinitely great number of laminæ, as in last Art., the centres of mass of these laminæ all lie on the right line joining the vertex to the centroid of the base. Hence the centre of mass of the whole pyramid lies on this line; and by what we have just proved, it must be one-fourth of the way up this line. There is no limit to the number of sides of the polygon; hence they may form a continuous curve.

Therefore—

The centre of mass of a cone whose base is any plane curve whatever is found by joining the centroid of the base to the vertex, and taking a point one-fourth of the way up this line.

164.] **Theorem.** *If the mass of each of a system of bodies be multiplied by the square of the distance of its centre of mass from a given point, the sum of the products thus obtained is least when the given point is the centre of mass of the system of bodies.*

This theorem, which is well known in elementary geometry, admits of a very simple analytical proof.

Let $(\bar{x}, \bar{y}, \bar{z})$ be the co-ordinates of the centre of mass, G, of the system with reference to rectangular axes through any point, O, and let (x_1, y_1, z_1), (x_2, y_2, z_2), ..., be the co-ordinates of the centres of mass, A_1, A_2, ..., of the bodies whose masses are $m_1, m_2, \ldots$. Then

$$GA_1^2 = (\bar{x}-x_1)^2+(\bar{y}-y_1)^2+(\bar{z}-z_1)^2. \quad (1)$$

Similarly,
$$GA_2^2 = (\bar{x}-x_2)^2+(\bar{y}-y_2)^2+(\bar{z}-z_2)^2, \quad (2)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Multiplying the sides of these equations by m_1, m_2, ..., and adding, we have

$$\Sigma(m\,.\,GA^2) = (\bar{x}^2+\bar{y}^2+\bar{z}^2)\,.\,\Sigma m - 2\bar{x}\,.\,\Sigma mx - 2\bar{y}\,.\,\Sigma my$$
$$- 2\bar{z}\,.\,\Sigma mz + \Sigma m(x^2+y^2+z^2). \quad (3)$$

Now (Art. 159),

$$\Sigma mx = \bar{x}\,.\,\Sigma m, \quad \Sigma my = \bar{y}\,.\,\Sigma m, \quad \Sigma mz = \bar{z}\,.\,\Sigma m.$$

Hence (3) becomes

$$\Sigma(m\,.\,GA^2) = \Sigma m(x^2+y^2+z^2) - (\bar{x}^2+\bar{y}^2+\bar{z}^2)\,.\,\Sigma m,$$

or
$$\Sigma(m\,.\,GA^2) = \Sigma(m\,.\,OA^2) - OG^2\,.\,\Sigma m, \quad (4)$$

from which equation it appears that $\Sigma(m\,.\,GA^2)$ is always less than $\Sigma(m\,.\,OA^2)$ by the quantity $OG^2\,.\,\Sigma m$.

It can be shown that, if r_{12} denote the distance between the centres of mass of the masses m_1 and m_2, and M the sum of all the masses, $M\,.\,\Sigma(m\,.\,GA^2) = \Sigma(m_1 m_2 r_{12}^2)$.

For, let the centre of mass, G, be taken as origin. Then, denoting the co-ordinates of the points A_1, A_2, ... with reference to G by (x_1', y_1', z_1'), (x_2', y_2', z_2'), ...,

$$M\Sigma(m\,.\,GA^2) = m_1(m_1+m_2+\ldots)(x_1'^2+y_1'^2+z_1'^2)$$
$$+ m_2(m_1+m_2+\ldots)(x_2'^2+y_2'^2+z_2'^2)+\ldots. \quad (5)$$

Also (Art. 159)

$$0 = m_1x_1'+m_2x_2'+\ldots$$
$$0 = m_1y_1'+m_2y_2'+\ldots$$
$$0 = m_1z_1'+m_2z_2'+\ldots.$$

Squaring both sides of each of these last three equations, adding the results together, and subtracting their sum from (5), we have

$$M\,.\,\Sigma(m\,.\,GA^2) = m_1m_2(\overline{x_1-x_2}^2+\overline{y_1-y_2}^2+\overline{z_1-z_2}^2)+\ldots$$
$$= m_1m_2r_{12}^2+\ldots$$
$$= \Sigma m_1m_2r_{12}^2.$$

Hence, from (4),

$$OG = \sqrt{\frac{\Sigma(m\,.\,OA^2)}{M} - \frac{\Sigma(m_1m_2r_{12}^2)}{M^2}},$$

under which form Lagrange expresses the distance of the centre of mass of a system of bodies from a given point (see *Mécanique Analytique*, p. 61).

Equation (4) can be employed to prove the well-known expression for the distance between the centres of the inscribed and circumscribed circles of a plane triangle, viz.:

$$D^2 = R^2 - 2Rr,$$

D being the distance between the centres, and r and R being their radii, respectively.

(Suppose a system of particles at the vertices, the mass of each being proportional to the opposite side. Their centre of mass is the centre of the inscribed circle. The remainder is left to the student as an exercise.)

EXAMPLES.

1. To find the position of the centre of mass of the frustum of a pyramid.

Let the frustum be formed by the removal of the pyramid $abcD$ (Fig. 220) from the whole pyramid $ABCD$; let h and H be the perpendicular heights of these pyramids, respectively; and let m and M denote their masses.

Now if the perpendicular distances of the centres of mass of the pyramid $ABCD$, the pyramid $abcD$, and the frustum, from the base ABC be denoted by z_1, z_2, and z, respectively, we have (Art. 159)

$$Mz_1 = mz_2 + (M-m)\,z. \tag{1}$$

But $z_1 = \frac{H}{4}$, $z_2 = \frac{h}{4} + H - h = H - \frac{3}{4}h$. Also the masses of the pyramids are to each other as the cubes of their heights; therefore (1) gives

$$\frac{H^4}{4} = h^3\left(H - \tfrac{3}{4}h\right) + (H^3 - h^3)\,z,$$

or

$$4\,(H^3 - h^3)\,z = H^4 - 4Hh^3 + 3h^4$$
$$= (H-h)^2\,(H^2 + 2Hh + 3h^2)$$
$$\therefore\quad z = \frac{(H-h)}{4} \cdot \frac{(H^2 + 2Hh + 3h^2)}{H^2 + Hh + h^2}. \tag{2}$$

Instead of the heights we can use the square roots of the areas of the bases, to which the heights are proportional. If these areas are denoted by A and a, we have

$$z = \frac{H-h}{4} \cdot \frac{A + 2\sqrt{Aa} + 3a}{A + \sqrt{Aa} + a}. \tag{3}$$

The centre of mass, G', of the frustum obviously lies on the line Nn (Fig. 220) between N and G; and (3) evidently gives

$$NG' = \frac{Nn}{4} \cdot \frac{A + 2\sqrt{Aa} + 3a}{A + \sqrt{Aa} + a}. \tag{4}$$

It is clear that the position of the centre of mass of the frustum of a cone standing on any plane base is also given by these equations.

2. To find the centre of mass of a board of uniform thickness and density whose figure is that of a quadrilateral.

Let $ABCD$ be the quadrilateral; draw the line AC, which divides the quadrilateral into two triangles; let L and M be the centroids of the triangles ABC and ADC, respectively; and let the line LM meet AC in N.

Then the centroid of the quadrilateral is a point, G, on LM such that $\dfrac{MG}{LG} = \dfrac{\text{area } ABC}{\text{area } ADC} = \dfrac{\text{area } ALC}{\text{area } AMC} = \dfrac{\text{perp. from } L \text{ on } AC}{\text{perp. from } M \text{ on } AC} = \dfrac{LN}{MN}$;

therefore $\dfrac{MG}{LM} = \dfrac{LN}{LM}$, or $MG = LN$.

The centre of mass is therefore found by taking a point, G, on LM, such that $MG = LN$.

Another construction. The student will find little difficulty in proving the following construction. Draw the diagonals AC and BD, meeting in the point O. On AC take a point C', such that $AC' = CO$, and on BD take a point B', such that $DB' = BO$. Then the centroid of the quadrilateral is the centroid of the triangle $B'OC'$.

3. From a triangular board of uniform thickness and density the portion constituting the area of the inscribed circle is removed; prove that the distance of the centre of mass of the remainder from any side (a) is

$$\frac{\Delta}{3as} \cdot \frac{2s^3 - 3\pi a\Delta}{s^2 - \pi\Delta},$$

Δ being the area, and s half the sum of the sides, of the board.

4. If a tetrahedron be formed by the centres of mass of any four masses, prove that each mass is proportional to the tetrahedron standing on the opposite face and having for vertex the common centre of mass of the masses.

5. If at the vertices of a triangle there be placed three masses each of which is proportional to the opposite side of the triangle, prove that their centre of mass is the centre of the circle inscribed in the triangle.

6. Prove that the centre of mass of a system of uniform bars forming a triangle is the centre of the circle inscribed in the triangle formed by the middle points of the bars.

7. A figure is formed by a right-angled triangle whose sides are

a, b, and c, and the squares constructed on these sides; find the distance of the centroid of this figure from the greatest side (c).

Ans. $\dfrac{ab}{3c} \cdot \dfrac{3c^2-5ab}{4c^2+ab}$.

8. Prove that the centroid of a trapezium divides the line joining the middle points of the two parallel sides in the ratio $\dfrac{a+2b}{2a+b}$, the lengths of these sides being a and b.

Prove also the following construction for the centroid :—

The vertices, in order, being A, B, C, D, and the parallel sides AB and CD, produce BA to A', and AB to B', so that $AA' = BB' = CD$; also produce DC to C', and CD to D', so that $CC' = DD' = AB$; then the point of intersection of $A'C'$ and $B'D'$ is the required centroid.

9. A right line passing through a fixed point intersects two fixed right lines; find the locus of the centroid of the triangle formed by the variable line and the two fixed lines.

Ans. If the co-ordinates of the fixed point with reference to the two fixed lines as axes are a and b, the locus is the hyperbola

$$(3x-a)(3y-b) = ab.$$

10. If the right line in the last example, instead of passing through a fixed point, cut off a triangle of constant area, find the locus of the centroid of the triangle.

Ans. If ω is the angle between the fixed lines, and k^2 the constant area, the locus is the hyperbola

$$9xy \sin\omega = 2k^2.$$

11. From a sphere of radius R is removed a sphere of radius r, the distance between their centres being c; find the centre of mass of the remainder.

Ans. It is on the line joining their centres, and at a distance $\dfrac{cr^3}{R^3-r^3}$ from the centre.

12. Every body has one and only one centre of mass. Hence show that the lines joining the middle points of the opposite sides of a quadrilateral bisect each other.

(Consider four equal particles at the vertices.)

13. From the vertices of a given triangle let perpendiculars be drawn to the opposite sides. Find the distances of the centroid of the triangle formed by the feet of these perpendiculars from the sides of the given triangle.

Ans. The distance from the side a is $\frac{1}{3} a \sin A \cos(B-C)$.

14. A thin uniform wire is bent into the form of a triangle ABC, and particles of weights, P, Q, R, are placed at the angular points A, B, C, respectively; prove that if the centre of mass of the particles coincides with that of the wire,

$$P:Q:R = b+c:c+a:a+b.$$

(Wolstenholme's *Book of Mathematical Problems.*)

15. Find the centroid of the triangle formed by the points in which the bisectors of the angles of a given triangle meet the opposite sides.

Ans. If Δ denote the area of the given triangle, whose sides are a, b, c, the distance of the centroid from the side a is

$$\tfrac{2}{3}\Delta\frac{2a+b+c}{(a+b)(a+c)}.$$

16. A uniform wire of given length is formed into a triangle of which one angle is given; find the locus of the centre of mass of the wire referred to the sides containing the given angle as axes.

Ans. If C is the given angle, and $4l$ the length of the wire, the locus is the ellipse

$$(l-x-y)^2+2(l-x-y)(2l-x-y)\sin^2\frac{C}{2}+4xy\sin^4\frac{C}{2}=0.$$

17. If particles be placed at the angular points of a tetrahedron, proportional respectively to the areas of the opposite faces, their centre of mass will be the centre of the sphere inscribed in the tetrahedron.

(Wolstenholme's *Book of Mathematical Problems.*)

18. Prove that the centroid of the surface of a tetrahedron is the centre of the sphere inscribed in the tetrahedron formed by joining the centroids of the faces.

19. If z_1, z_2, z_3, z_4 are the distances (rectangular or oblique) of the vertices of any quadrilateral area from a plane, and ζ the distance of the point of intersection of its diagonals from the plane, the distance of its centroid from the plane is

$$\tfrac{1}{3}(z_1+z_2+z_3+z_4-\zeta).$$

20. On the base of a solid homogeneous hemisphere is constructed a solid cylinder; determine the height of the latter so that the centre of mass of the compound body may be the centre of the sphere, being given that the centre of mass of a solid hemisphere is $\frac{3}{8}r$ from the centre, where r is the radius.

Ans. $\frac{1}{2}r\sqrt{2}$.

21. A hollow thin metallic cone without a base is held with its axis vertical and vertex down; the radius of the base is 5 inches, the height 12, the thickness $\frac{1}{26}$ inch, and the specific weight of the metal 8 times that of water. If the cone is filled to the height of 6 inches with water, find the position of the centre of gravity of the water and vessel.

Ans. It is about 6·6 inches from the vertex.

SECTION II.

Centres of Mass in General.

165.] **Rule.** The general formulae of Article 159, for the co-ordinates of the centre of mass of a quantity of matter arranged in any manner, assume particular forms according as the matter is arranged in the form of a wire of any shape, an area or thin lamina of any shape, or a solid. Then, again, they assume particular forms in each of these cases according to the manner in which the matter is supposed to be divided into elementary portions.

Many students are in the habit of remembering a special formula for each of these numerous cases; such a habit, however, is not only useless but injurious. It is much better to consider the fundamental formula (1) of Art. 159, or the method of p. 131, as furnishing the following Rule which covers all possible cases:

Divide the quantity of matter, in any way, into elementary portions; find the position of the centre of mass of each of these portions; then multiply the mass of each portion by the co-ordinate of its centre of mass, and take the integral of this product; and finally divide this integral by the whole quantity of matter. The result is the co-ordinate of the centre of mass required.*

166.] **Centre of Mass of the Arc of a Curve.** If the matter whose centre of mass we desire to find is arranged in the shape of the arc of any curve, the co-ordinates of its centre of mass are obtained from the formulae of Art. 159, in which dm now denotes the mass of an elementary length of the curve.

Let ds denote the length of an elementary portion of the curve contained between two points, P and Q (Fig. 221); let k denote the mean area of a section of the curve between P and Q; and let ρ denote the density of the matter in the neighbourhood of P and Q. Then, since the quantity of matter in any space is equal to the product of the volume and the density, the quantity of matter between P and Q is $k\rho\, ds$.

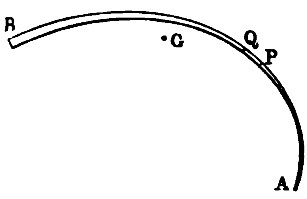

Fig. 221.

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* The co-ordinates are supposed to be such as are measured parallel to a given line. The rule would not hold if by *co-ordinate* were understood *polar co-ordinate* for instance.

Again, the centre of mass of this element is evidently the middle point of PQ.

And since to obtain G, the centre of mass of the whole mass, the co-ordinates of this middle point must be multiplied by the infinitesimal $k\rho ds$, the co-ordinates of the centre of mass of PQ may be taken to be the same as those of P.

Replacing dm in the general formulae by the linear element $k\rho ds$, we obtain for the position of the centre of mass of matter arranged in the form of any curve the equations

$$\bar{x} = \frac{\int k\rho x ds}{\int k\rho ds},$$

$$\bar{y} = \frac{\int k\rho y ds}{\int k\rho ds},$$

$$\bar{z} = \frac{\int k\rho z ds}{\int k\rho ds}.$$

The quantities k and ρ must be given as functions of the position of the point P before the integrations can be performed.

EXAMPLES.

1. To find the position of the centroid of a circular arc of uniform thickness and density.

Let AB be the arc, M its middle point, and O the centre of the circle. Then it is manifest from symmetry that the centroid must lie on the line OM. Take OM as axis of x. Then, since k and ρ are constant, we have

$$\bar{x} = \frac{\int x ds}{\int ds},$$

x being the co-ordinate of any point, P, in the arc. Let θ be the angle POM and a the radius of the circle. Then

$$x = a\cos\theta, \text{ and } ds = a d\theta.$$

Hence

$$\bar{x} = a\frac{\int \cos\theta d\theta}{\int d\theta},$$

the integration to be extended over the whole arc. Now if the angle $BOA = 2\alpha$, the integration must be taken from $\theta = -\alpha$ to $\theta = \alpha$. Therefore

$$\bar{x} = a\frac{\sin\alpha}{\alpha}.$$

Hence *the distance of the centroid of the arc of a circle from the centre is the product of the radius and the chord of the arc divided by the length of the arc.*

The distance of the centroid of a semicircle from the centre is $\frac{2a}{\pi}$.

2. Find the centre of mass of a circular arc of uniform section, the density varying as the length of the arc measured from one extremity.

Let AB be the arc; let the density at any point $P = \mu . AP$, and let OA be taken as axis of x. Then if $\angle AOB = \alpha$, and $AP = s$, we have

$$\bar{x} = \frac{\int sxds}{\int sds} = a\frac{\int_0^\alpha \theta\cos\theta d\theta}{\int_0^\alpha \theta d\theta}$$

$$= 2a\frac{\alpha\sin\alpha + \cos\alpha - 1}{\alpha^2}.$$

Similarly,
$$\bar{y} = \frac{\int syds}{\int sds} = a\frac{\int_0^\alpha \theta\sin\theta d\theta}{\int_0^\alpha \theta d\theta}$$

$$= 2a\frac{\sin\alpha - \alpha\cos\alpha}{\alpha^2}.$$

3. One extremity, A, of the arc, AB, of a curve being fixed, while the other extremity, B, varies, it is required to construct at any point the tangent to the locus of the centroid of the variable arc AB.

Let AB be a portion of the arc of any curve, and let G be the centroid of AB. Then if B' be a point on the given curve very close to B, the centroid of the whole arc AB' is obtained by joining the centroid, G, of AB to the centroid of BB', and dividing the joining line inversely as the lengths of AB and BB'. But the centroid of BB' is its middle point. Hence the centroid of AB' lies on the line joining G to the middle point BB'. In the limit, therefore, the line joining G to its next consecutive position is the line GB, which is, then, the tangent at G to the locus of G.

4. Find the position of the centroid of the arc of a semi-cardioid.

Ans. The equation of the curve being $r = a(1+\cos\theta)$, the co-ordinates of its centroid referred to the axis of the curve and a perpendicular line through the cusp as axes of x and y are

$$\bar{x} = \bar{y} = \frac{4}{5}a.$$

5. Find the equation of the line joining the centroid of the arc of half a loop of a lemniscate to the double point.

Ans. The axes of x and y being the axis of the curve and a perpendicular line, the equation of the required line is

$$y = (\sqrt{2}-1).x.$$

6. Find the centroid of the arc of a semi-cycloid.

Ans. The axis of x being a tangent at the vertex, and a the radius of the generating circle,

$$\bar{x} = \left(\pi - \frac{4}{3}\right)a, \quad \bar{y} = \frac{2}{3}a.$$

7. Find the distance of the centroid of the catenary

$$y=\frac{c}{2}(e^{\frac{x}{c}}+e^{-\frac{x}{c}})$$

from the axis of x, the curve being divided into two equal portions by the axis of y.

Ans. If $2l$ is the length of the curve and k the ordinate of its extremity, the centroid lies on the axis of y at a distance $\frac{kl+ac}{2l}$ from the axis of x.

8. Find a law of density of a wire of uniform section bent into the shape of a cycloid so that its centre of mass shall be half way up its axis.

Ans. If the density varies as the length of the arc measured from the vertex, the result will follow.

9. If the density of a cycloidal arc varies as the n^{th} power of the arc measured from the vertex, find the position of the centre of mass of the curve.

Ans. On the axis at a distance $2\frac{n+1}{n+3}a$ from the vertex, a being the radius of the generating circle.

10. One extremity of a circular arc is fixed while the other varies along the circle; trace the locus of the centroids of the varying arcs, and prove that the *algebraic* sum of the intercepts of the locus on the diameter perpendicular to that passing through the fixed extremity of the arcs is equal to half the radius.

167.] **Centroid of a Plane Area.** Let $APQB$ (Fig. 222) be any curve whose equation is given, and let it be required to find the centroid of the area, $CABD$, of a lamina included between a given portion, AB, of the curve, two extreme ordinates, AC and BD, and the axis of x, the lamina being supposed of uniform thickness and density. In accordance with the rule of Art. 165, we break up the area into elementary portions. Suppose that this is done by taking rectangular strips, such as $PQNM$, included between two very close ordinates, PM and QN, and let g be the centre of mass of this strip.

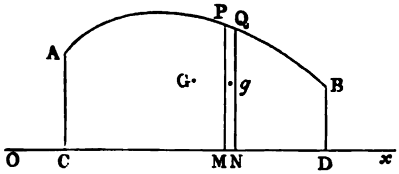

Fig. 222.

Let the co-ordinates of P be (x, y) and those of Q $(x+dx, y+dy)$; let ρ be the density and k the thickness of the lamina.

Then the mass, dm, of the rectangular strip is

$$k\rho y dx.$$

Also the co-ordinates of g are $\left(x+\epsilon, \frac{y}{2}+\epsilon'\right)$, ϵ and ϵ' being extremely small quantities of the same order of magnitude as dx and dy.

Following the rule of Art. 165, to obtain the abscissa of G, the centroid of the area, we shall have to take the integral of the product $$k\rho y(x+\epsilon)dx.$$

Now ϵdx is an infinitesimal of the second order, and is therefore to be neglected in the integral. Hence if $\bar{x}$ and $\bar{y}$ are the co-ordinates of G, we have evidently, since k and ρ are constants,

$$\bar{x} = \frac{\int xy dx}{\int y dx}, \quad \bar{y} = \tfrac{1}{2}\frac{\int y^2 dx}{\int y dx},$$

the integrations extending over the whole area $CABD$.

Examples.

1. Find the centroid of the area of a semi-cycloid.

Taking the line joining the extremities of the arc of the whole curve as axis of x, and a perpendicular through the vertex as axis of y, the curve is given by the equations

$$x = a(\theta+\sin\theta),$$
$$y = a(1+\cos\theta).$$

Hence $y dx = 4a^2 \cos^4\frac{\theta}{2} d\theta$, and we have

$$\bar{x} = a\frac{\int_0^\pi (\theta+\sin\theta)\cos^4\frac{\theta}{2} d\theta}{\int_0^\pi \cos^4\frac{\theta}{2} d\theta}, \quad \bar{y} = a\frac{\int_0^\pi \cos^6\frac{\theta}{2} d\theta}{\int_0^\pi \cos^4\frac{\theta}{2} d\theta}.$$

To find $\int_0^\pi \theta\cos^4\frac{\theta}{2} d\theta$, write it

$$\frac{1}{4}\int_0^\pi \theta(1+\cos\theta)^2 d\theta, \text{ or } \frac{1}{4}\int_0^\pi \theta\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta.$$

Now $\int\theta\cos n\theta d\theta = \frac{n\theta\sin n\theta + \cos n\theta}{n^2}$. Hence the integral in question $= \frac{3\pi^2-16}{16}$.

Again, $\int_0^\pi \sin\theta\cos^4\frac{\theta}{2} d\theta = 2\int_0^\pi \sin\frac{\theta}{2}\cos^5\frac{\theta}{2} d\theta = \frac{2}{3}$.

Hence $$\bar{x} = \frac{9\pi^2 - 16}{18\pi} \cdot a.$$

And evidently $$\bar{y} = \frac{5}{6} a.$$

2. If the ordinates of a given curve, U, be all diminished or increased in a given ratio and a new curve, U', thus formed, prove that the centroid of any portion of U' cut off by a right line is obtained by diminishing or increasing in the same ratio the ordinate of the centroid of the corresponding portion of U.

Let one line parallel to the axis of y meet U and U' in P and P' respectively, and let another such line meet them in Q and Q'. Draw the right lines PQ and $P'Q'$; then these lines cut off corresponding portions of the two curves. From any point, M, on U draw a line parallel to the axis of y meeting the right line PQ in N, and U' and $P'Q'$ in M' and N', respectively. Denote the ordinates of M and N by y and z; then it is clear that if k is the number by which the ordinates of U are multiplied to obtain those of U', the ordinates of M' and N' are ky and kz, respectively. All these points have a common abscissa, x. An ordinate drawn with the abscissa $x + dx$ includes with the ordinate $MNM'N'$, the curve U, and the line PQ a strip of area equal to $(y-z)\,dx$, while the corresponding strip of the area of U' cut off by $P'Q'$ is $k(y-z)\,dx$. Again, the ordinate of the middle point of the first strip is $\frac{y+z}{2}$, and that of the middle point of the second strip is $k\frac{y+z}{2}$.

Hence if $\bar{y}$ and $\bar{y}'$ denote the ordinates of the centroids of the portions of U and U' cut off by PQ and $P'Q'$, respectively,

$$\bar{y}' = \frac{1}{2}\frac{\int k^2 (y^2 - z^2)\,dx}{\int k(y-z)\,dx}$$

$$= k \cdot \bar{y}.$$

Let PQ cut off in all positions a constant area from U; then it is evident that $P'Q'$ cuts off a constant area from U'. Suppose, moreover, that in this case the locus of the centroid of the portion of U is a curve whose equation is $f(x, y) = 0$;

then clearly the locus of the centroid of the corresponding portion of U' of constant area cut off by a right line is the curve

$$f\left(x, \frac{y}{k}\right) = 0.$$

If the lines PQ and $P'Q'$ are replaced by two curves, the second of which is deduced from the first as U' was from U, the same results evidently follow.

3. Find the centroid of a quadrant of an ellipse.

Ans. $\bar{x} = \frac{4a}{3\pi}$, $\bar{y} = \frac{4b}{3\pi}$.

4. A right line cuts off a constant area from an ellipse; find the locus of the centroid of the portion cut off.

Ans. An ellipse concentric and coaxal with the given one.

5. Find the centroid of a quadrant of the curve $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$.

$$Ans.\ \bar{x}=\frac{2.4.6.8}{3.5.7.9}\cdot\frac{2a}{\pi};\quad \bar{y}=\frac{2.4.6.8}{3.5.7.9}\cdot\frac{2b}{\pi}.$$

(Assume $x=a\cos^3\phi,\ y=b\sin^3\phi$.)

6. Find the centroid of any segment of a parabola cut off by a right line.

Ans. On the diameter conjugate to the given line at a distance from the curve equal to $\frac{3}{5}$ of the portion of the diameter intercepted by the given line.

7. Through a given point, O, is drawn a fixed right line meeting a curve in A; through O is also drawn another right line meeting the curve in P. It is required to construct at any point the tangent to the locus described by the centroid of the area AOP as the line OP varies.

Ans. Let G be the centroid of AOP, and take a point Q on OP such that $OQ=\frac{2}{3}OP$. Then GQ is the tangent to the locus at G. (See Example 3, p. 309.)

8. Find the centroid of a semi-ellipse cut off by any diameter.

Ans. It is on the diameter conjugate to the given one and at a distance $\frac{4a'}{3\pi}$ from the centre, $2a'$ being the length of this conjugate diameter.

9. Find the centroid of the area included by a parabola and two tangents.

Ans. If a and b are the lengths of the tangents (which are taken as axes of x and y), $\bar{x}=\frac{a}{5}$, $\bar{y}=\frac{b}{5}$.

(The equation of the parabola is $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1$. Assume

$$x=a\cos^4\phi,\ y=b\sin^4\phi.)$$

The particular manner in which it is advisable to break up the area whose centroid is required varies with the nature of the area itself. Thus, let the area be that included between the axis of x and two curves, AC and BC (Fig. 223), whose equations are given. In this case the area may be broken up into thin strips, such as $PQQ'P'$, parallel

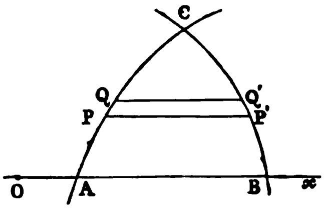

Fig. 223.

to the axis of x. Let (x, y) be the co-ordinates of P and (x', y) those of P'. Then the area of the strip is $(x'-x)dy$, and the co-ordinates of its centroid are $\frac{1}{2}(x'+x)$ and y. Hence if no portion of the area considered is above a parallel to Ox drawn through C, the co-ordinates of its centroid are given by the equations

$$\bar{x} = \tfrac{1}{2}\frac{\int(x'^2-x^2)dy}{\int(x'-x)dy}, \quad \bar{y} = \frac{\int y(x'-x)dy}{\int(x'-x)\,dy},$$

in which the limits of y are 0 and the ordinate of C. The values of x' and x are of course given in terms of y from the equations of the two curves.

For example, let it be required to find the centroid of the area included between a parabola and a circle described with the vertex of the parabola as centre and a radius equal to $\frac{3}{8}$ of its latus rectum. The centroid is on the axis of the parabola. Let the equation of the parabola be $y^2 = 4mx$; then the equation of the circle is $x^2+y^2=\frac{9}{4}m^2$; and the ordinate of C, their point of intersection, is $m\sqrt{2}$.

Hence

$$\bar{x} = \tfrac{1}{2}\frac{\int_0^{m\sqrt{2}}\left(\frac{9}{4}m^2-y^2-\frac{y^4}{16m^2}\right)dy}{\int_0^{m\sqrt{2}}\left(\sqrt{\frac{9}{4}m^2-y^2}-\frac{y^2}{4m}\right)dy}$$

$$= \frac{36m}{16+27\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)},$$

as the student will find without much difficulty.

EXAMPLES.

1. Find the centroid of the area included between the arc of a semi-cycloid, the circumference of the generating circle, and the line joining the extremities of the cycloid.

Ans. The common tangent to the circle and cycloid at the vertex of the latter being taken as axis of x, the vertex being origin, and a the radius of the generating circle,

$$\bar{x} = \frac{3\pi^2-8}{4\pi}\cdot a; \quad \bar{y} = \frac{5}{4}a.$$

2. Find the locus of the centroid of the area of a parabola cut off by a variable right line drawn through the vertex.

Ans. If $4m$ is the latus rectum of the parabola, the locus is another parabola whose equation is $y^2 = \frac{5}{2}mx$.

(The student may verify the construction of Example 7, p. 313, for the tangent to this locus.)

3. Find the centroid of the portion of an ellipse cut off by a line joining the extremities of the major and minor axes.

Ans. $\bar{x} = \frac{2}{3} \cdot \frac{a}{\pi - 2}$; $\bar{y} = \frac{2}{3} \cdot \frac{b}{\pi - 2}$.

168.] **Graphic Construction of the Centroid of a Plane Area.** The following method of determining the centroid of any plane area is taken from Collignon's *Statique*, p. 315.

Let $APBQ$ be any plane area, and let Ox be any line in its plane. Then, if the distances of the centroid from Ox and any other line in the plane are known, the position of the point is known.

Draw any line, $O'x'$, parallel to Ox (axis of x) in the plane of the curve, and let the perpendicular distance between Ox and $O'x'$ be a. Let the area be broken up into narrow rectangular strips, such as $PP'Q'Q$, by lines parallel to the axis of x. Then if $PQ = z$, the area of the strip $= z\,dy$, the distance of PQ from Ox being y.

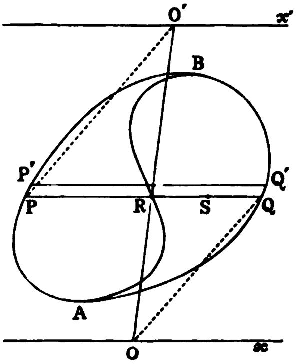

Fig. 224.

Hence the distance, $\bar{y}$, of the centroid of the area from Ox is given by the equation

$$\bar{y} = \frac{\int yz\,dy}{\int z\,dy} = \frac{\int yz\,dy}{A_1}, \tag{1}$$

A_1 being the area of the figure, and the values of y running from the ordinate of A to that of B, at which points the tangents are parallel to Ox. Now take any point, O, on Ox; draw OQ, and draw PO' parallel to OQ. Let the line OO' meet PQ in R. Then by similar triangles

$$\frac{QR}{RP} = \frac{OR}{RO'}; \quad \therefore \quad \frac{QR}{PQ} = \frac{OR}{OO'},$$

or, z' denoting the length QR,

$$az' = yz. \tag{2}$$

Let the locus of R corresponding to all strips of the given area be constructed. It will be a curve, ARB, passing through the points A and B.

Substituting the value of yz from (2) in (1), we have

$$\bar{y} = \frac{a\int z'dy}{A_1},$$

in which the limits of y are the same as before. But $\int z'dy$ is the area, A_2, between the curves ARB and AQB. Hence

$$\bar{y} = a\frac{A_2}{A_1}.$$

The distance of the centroid from Ox is therefore known. Similarly its distance from any other line can be found, and therefore the position of the point is determined.

If a point S is deduced from R in the same way as that in which R was deduced from P, and if $QS = z''$, we shall have as before

$$az'' = z'y = \frac{y^2z}{a}.$$

If therefore the locus of S is constructed, the area included between it and AQB multiplied by a^2 will be the value of the integral $\int y^2z\,dy$ extended over the original area.

By the construction of successive curves such as ARB we represent the values of $\int y^3z\,dy$, $\int y^4z\,dy$, &c., graphically.

An ingenious instrument founded on these principles—the Integrometer of M. Deprez—is described by Collignon in the *Annales des Ponts et Chaussées* for March, 1872.

EXAMPLE.

In finding by this method the centroid of a portion of a parabola cut off by a double ordinate at a distance h from the vertex, prove that if the tangent at the vertex and the given double ordinate are taken as the lines Ox and $O'x'$, the equation of the curve ARB will be

$$h^2y^2 = 4mx(h-2x)^2.$$

This curve (both branches being drawn) has a loop between the values $x = 0$ and $x = \frac{1}{2}h$, and passes through the extremities of the double ordinate.

169.] **Polar Elements of a Plane Area.** Let it be required to find the centroid of a portion of a plane area or thin lamina bounded by a portion of any curve, *AB* (Fig. 225), and by two extreme radii vectores, *OA* and *OB*, drawn through a given point, *O*. It is obvious that in this case it is advisable in applying the rule of Art. 165 to decompose the area into triangular strips, such as *POQ*, included between two very close radii vectores. If $OP = r$, and $\angle POx = \theta$, the element of area, *POQ*, is equal to

$$\tfrac{1}{2} r^2 d\theta;$$

and if the thickness and density of the lamina are uniform, the centre of mass of this element is a point g which may be considered as on OP at a distance $\tfrac{2}{3} r$ from O.

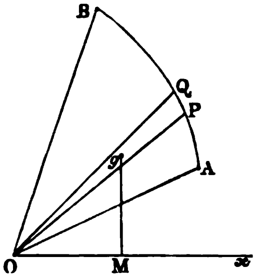

Fig. 225.

Hence if Ox is the axis of x, the co-ordinates of g are ultimately

$$\tfrac{2}{3} r \cos\theta, \quad \text{and} \quad \tfrac{2}{3} r \sin\theta.$$

Applying the rule of Art. 165, we then have

$$\bar{x} = \tfrac{2}{3}\frac{\int r^3 \cos\theta\, d\theta}{\int r^2 d\theta}; \quad \bar{y} = \tfrac{2}{3}\frac{\int r^3 \sin\theta\, d\theta}{\int r^2 d\theta}.$$

For example, to find the centroid of a loop of Bernoulli's Lemniscate whose equation is $r^2 = a^2 \cos 2\theta$.

The axis of the loop being taken as axis of x, the abscissa of the centroid of the whole loop is evidently the same as that of the half loop above the axis;

$$\therefore\ \bar{x} = \frac{2a}{3}\frac{\int_0^{\frac{\pi}{4}} \cos^{\frac{3}{2}} 2\theta \cos\theta\, d\theta}{\int_0^{\frac{\pi}{4}} \cos 2\theta\, d\theta}$$

$$= \frac{4a}{3}\int_0^{\frac{\pi}{4}} (1 - 2\sin^2\theta)^{\frac{3}{2}} . d\sin\theta.$$

Putting $\sin\theta = \frac{\sin\phi}{\sqrt{2}}$, this integral becomes

$$\frac{4a}{3\sqrt{2}}\int_0^{\frac{\pi}{2}} \cos^4\phi\, d\phi,$$

which $= \frac{4a}{3\sqrt{2}} \cdot \frac{1.3}{2.4} \cdot \frac{\pi}{2}$. Therefore

$$\bar{x} = \frac{\pi a}{4\sqrt{2}}.$$

EXAMPLES.

1. To find the centroid of a given sector of a circle.

Ans. It is on the diameter bisecting the arc, at a distance from the centre equal to $\frac{2}{3}$ of the product of the radius and the chord of the arc divided by the length of the arc.

2. Find the centroid of a portion of an equiangular spiral included by the initial line and a given radius vector.

Ans. The initial line being taken as axis of x, the equation of the spiral being $r = ae^{k\theta}$, and a being the angle of the given radius vector,

$$\bar{x} = \frac{4ka}{3(1+9k^2)} \cdot \frac{e^{3ka}\sin a + 3ke^{3ka}\cos a - 3k}{e^{2ka}-1};$$

$$\bar{y} = \frac{4ka}{3(1+9k^2)} \cdot \frac{1 - e^{3ka}\cos a + 3ke^{3ka}\sin a}{e^{2ka}-1}.$$

3. When $a = 0$, in the preceding question, find the values of $\bar{x}$ and $\bar{y}$, and explain the result.

4. Find the centroid of the portion of a parabolic area included between the axis and a radius vector drawn through the focus.

Ans. If $4m$ is the latus rectum, and t the tangent of half the angle between the given radius vector and the axis,

$$\bar{x} = \frac{2m}{3} \cdot \frac{1-\frac{1}{5}t^4}{1+\frac{1}{3}t^2}; \quad \bar{y} = \frac{2m}{3} \cdot \frac{t+\frac{1}{2}t^3}{1+\frac{1}{3}t^2}.$$

170.] **Double Integration.** When the density of the lamina varies from point to point it may be necessary to divide it into infinitesimal portions of the second order instead of strips (triangular or rectangular) whose areas are infinitesimals of the first order.

Thus, suppose that the lamina AOB (Fig. 225) is not of uniform density. Then if we break it up into triangular strips, such as POQ, the element of mass will be no longer proportional to the area POQ or $\frac{1}{2}r^2d\theta$; and, moreover, the centre of mass of the strip will not be $\frac{2}{3}r$ distant from O.

Let a series of circles be described round O as centre, the distance between two successive circles of the series being dr'. These circles will divide the strip POQ into an indefinitely great number of rectangular elements; and if one of these is included between the circles of radii r' and $r'+dr'$, its area will be

$$r'dr'd\theta.$$

If ρ is the density and k the thickness of the lamina at this element, the element of mass will be

$$k\rho r' dr' d\theta.$$

Also the rectangular co-ordinates of the centre of mass of the element are ultimately $r'\cos\theta$ and $r'\sin\theta$.

Now to find the abscissa of the centre of mass we must perform the summations $\int x\,dm$ and $\int dm$ over the whole area considered.

The contribution to the first of these summations given by the strip POQ is evidently

$$\cos\theta\, d\theta\int_0^r k\rho r'^2 dr';$$

and the contribution to the second is

$$d\theta\int_0^r k\rho r' dr'.$$

In each of these latter integrals the values k and ρ in terms of r' and θ must be substituted, and the integrations are to be performed on the supposition that θ is constant while r' runs from 0 to r.

The quantity $\cos\theta\, d\theta\int_0^r k\rho r'^2 dr'$ will then assume the form $\phi(r,\theta)\,.\cos\theta\, d\theta$. But since the curve AB is given, r is given as a function of θ. Hence this quantity assumes the form $f(\theta)\,.\cos\theta\, d\theta$, which is the final form of the contribution of the strip POQ. If we wish to find how much is contributed by all the strips of the area, we must integrate $f(\theta)\,.\cos\theta\, d\theta$ from $\theta = AOx$ to $\theta = BOx$.

This double process of integration—first with regard to r', and then with regard to θ—is expressed by the symbols of double integration thus:—

$$\int x\,dm = \int_\alpha^\beta\int_0^r k\rho r'^2\cos\theta\, dr' d\theta,$$

α and β denoting the angles AOx and BOx.

Hence we obtain

$$\bar{x} = \frac{\int_\alpha^\beta\int_0^r k\rho r'^2\cos\theta\, dr' d\theta}{\int_\alpha^\beta\int_0^r k\rho r' dr' d\theta}; \quad \bar{y} = \frac{\int_\alpha^\beta\int_0^r k\rho r'^2\sin\theta\, dr' d\theta}{\int_\alpha^\beta\int_0^r k\rho r' dr' d\theta}.$$

Let it be required, for example, to find the centroid of the area of a cardioid in which the density at a point varies as the n^{th} power of the distance of the point from the cusp.

Here $\rho = \mu r'^n$, and k is constant; therefore, the abscissa being the same for the whole curve as for the half above the axis,

$$\bar{x} = \frac{\int_0^\pi \int_0^r r'^{n+2} \cos\theta \, dr' \, d\theta}{\int_0^\pi \int_0^r r'^{n+1} dr' \, d\theta}.$$

Integrating first with regard to r', we have

$$\frac{n+2}{n+3} \frac{\int_0^\pi r^{n+3} \cos\theta \, d\theta}{\int_0^\pi r^{n+2} d\theta}.$$

But $r = 2a \cos^2 \frac{\theta}{2}$. Substituting this value and putting $\frac{\theta}{2} = \phi$, we have

$$\frac{2(n+2)}{n+3} a . \frac{\int_0^{\frac{\pi}{2}} \cos^{2n+6}\phi (2\cos^2\phi - 1) d\phi}{\int_0^{\frac{\pi}{2}} \cos^{2n+4}\phi \, d\phi}.$$

These definite integrals are well known. Dividing the numerator and denominator by $\frac{1.3.5 \ldots 2n+3}{2.4.6 \ldots 2n+4} \cdot \frac{\pi}{2}$, we have

$$\bar{x} = \frac{2(n+2)}{n+3} \left\{ 2 . \frac{(2n+5)(2n+7)}{(2n+6)(2n+8)} - \frac{2n+5}{2n+6} \right\} a$$

$$= \frac{(n+2)(2n+5)}{(n+3)(n+4)} \cdot a.$$

The centroid evidently lies on the axis of symmetry, or $\bar{y} = 0$.

EXAMPLES.

1. Find the centre of mass of a circular sector in which the density varies as the n^{th} power of the distance from the centre.

Ans. $\frac{n+2}{n+3} \cdot \frac{ac}{l}$, where a is the radius of the circle, l the length of the arc, and c the length of the chord, of the sector.

2. Find the position of the centre of mass of a circular lamina in which the density at any point varies as the n^{th} power of the distance from a given point on the circumference.

Ans. It is on the diameter passing through the given point at a distance from this point equal to $\frac{2(n+2)}{n+4} a$, a being the radius.

Methods of double integration are also often employed when the elements of area are expressed in Cartesian co-ordinates. In this case, let the element of area at a point P, whose co-ordinates are (x', y'), be a small rectangle included between two very close lines parallel to the axis of x and two very close lines parallel to the axis of y. Then the element of area will be $dx'dy'$; and if ρ and k are the density and thickness of the lamina at the element, the element of mass,

$$dm = k\rho\, dx'dy'.$$

Also the co-ordinates of the centre of mass of this element are ultimately x' and y'. Hence

$$\bar{x} = \frac{\iint k\rho x'\,dx'\,dy'}{\iint k\rho\,dx'\,dy'}; \quad \bar{y} = \frac{\iint k\rho y'\,dx'\,dy'}{\iint k\rho\,dx'\,dy'}.$$

A single example will suffice to illustrate this method.

Let it be required to find the centre of mass of a quadrant of an ellipse included by the semi-axes, the density at any point being proportional to the product of the co-ordinates of this point.

Here $\rho = \mu . x'y'$, and since k is supposed constant,

$$\bar{x} = \frac{\iint x'^2 y'\,dx'dy'}{\iint x'y'\,dx'dy'}; \quad \bar{y} = \frac{\iint x'y'^2\,dx'dy'}{\iint x'y'\,dx'dy'}.$$

Let the integrations be performed first over a strip parallel to the axis of y. Then we integrate with respect to y', regarding x' as constant, from $y' = 0$ to $y' = y$, the ordinate of a point on the ellipse.

Hence $$\bar{x} = \frac{\int x'^2 y^2\,dx'}{\int x' y^2\,dx'}.$$

Here we must substitute the value of y in terms of x', and thus we get

$$\bar{x} = \frac{\int x'^2 (a^2 - x'^2)\,dx'}{\int x' (a^2 - x'^2)\,dx'},$$

in which summations the abscissa x' is to receive all values from 0 to a.

We easily obtain $\frac{8}{15}a$ and $\frac{8}{15}b$ for the co-ordinates of the centre of mass.

Cases may occur in which, although the density of the lamina varies from point to point, the process of double integration can be avoided by the judicious selection of an element of area.

Let it be required to find the centre of mass of a lamina in the form of a quadrant of an ellipse in which the density at any point varies as the distance of the point from the axis major.

Here, by dividing the area into rectangular strips parallel to the axis major, we obtain infinitesimal elements of the *first* order throughout each of which the density is constant. Hence our equations are

$$\bar{x} = \tfrac{1}{2}\frac{\int x^2 y\,dy}{\int xy\,dy}; \quad \bar{y} = \frac{\int xy^2 dy}{\int xy\,dy}.$$

Making the usual eccentric angle substitutions for x and y, we find

$$\bar{x} = \frac{3}{8}a, \quad \bar{y} = \frac{3\pi}{16}b.$$

171.] **Centroid of a Surface of Revolution.** Let a plane curve AB (Fig. 222) revolve round a line Ox (taken as axis of x) and generate a surface. Then the revolution of the elementary arc PQ ($= ds$) generates a portion of surface whose area is $2\pi y\,ds$; and if ρ is the density of the matter in this zone and k its thickness, the element of mass is $2\pi k\rho y\,ds$. Also the centre of mass of the zone is ultimately the point M, whose abscissa is x. Hence the centroid of the surface generated (which obviously lies on the axis of revolution) is at a distance from O given by the equation

$$\bar{x} = \frac{\int k\rho xy\,ds}{\int k\rho y\,ds},$$

the integrations being extended over the whole length of the generating curve.

For example, to find the centroid of the surface of a semi-ellipsoid of revolution round the minor axis, the density of any zone being proportional to its distance from the equatoreal plane, and the thickness being constant:—

The area of a zone at a distance y from the equatoreal plane being $2\pi x\,ds$, the position of the centroid is given by the equation

$$\bar{y} = \frac{\int xy^2 ds}{\int xy\,ds},$$

the integration extending over the arc of a quadrant of the generating ellipse. Using the eccentric angle, we have

$$x = a\cos\phi, \quad y = b\sin\phi, \quad ds = \sqrt{a^2\sin^2\phi + b^2\cos^2\phi}\,.\,d\phi,$$

a and b being the semi-axes of the ellipse.

Hence

$$\bar{y} = b\frac{\int_0^{\frac{\pi}{2}} \cos\phi\sin^2\phi\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}\,.\,d\phi}{\int_0^{\frac{\pi}{2}} \cos\phi\sin\phi\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}\,.\,d\phi}.$$

To find the integral in the numerator, put t for $\sin\phi$, and it becomes

$$\int_0^1 t^2\sqrt{b^2+c^2t^2}\,dt,$$

where $a^2-b^2=c^2$. This, again, is equal to

$$\frac{1}{c^2}\int_0^1 (b^2+c^2t^2-b^2)\sqrt{b^2+c^2t^2}\,.\,dt,$$

which $$=\frac{1}{c^2}\int_0^1 (b^2+c^2t^2)^{\frac{3}{2}}\,dt-\frac{b^2}{c^2}\int_0^1 (b^2+c^2t^2)^{\frac{1}{2}}\,dt;$$

and this, by making the first integral depend on the second, is easily proved to be

$$\left\{\frac{t(b^2+c^2t^2)^{\frac{3}{2}}}{4c^2}\right\}_0^1-\frac{b^2}{4c^2}\,.\int_0^1 (b^2+c^2t^2)^{\frac{1}{2}}\,dt.$$

The integral in this expression is one of the elementary forms in the Integral Calculus. Hence the numerator is

$$\frac{1}{8c^3}(2a^3c-ab^2c-b^4\log\frac{a+c}{b}).$$

The integral in the denominator is evidently

$$\tfrac{1}{2}\int_0^{\frac{\pi}{2}}\sqrt{b^2+c^2\sin^2\phi}\,.\,d\sin^2\phi,$$

which is equal to $\frac{1}{3c^2}(a^3-b^3)$.

Therefore $$\bar{y}=\frac{3b}{8}\cdot\frac{2a^3c-ab^2c-b^4\log\frac{a+c}{b}}{c(a^3-b^3)}.$$

For a sphere of radius a the value of $\bar{y}$ is easily proved by direct calculation to be $\frac{2}{3}a$; and the student may exercise himself in the evaluation of indeterminate forms by deducing this from the value of $\bar{y}$ given above. (For this purpose it will be advisable to put $\log\frac{a+c}{b}$ into the form $\frac{1}{2}\log\frac{a+c}{a-c}$, and expand.)

172.] Centroid of any Portion of a Spherical Surface. Let dS denote any portion of a spherical surface, and let $d\Sigma$ denote its projection on any plane passing through the centre of the sphere. Then, if this plane be taken as that of xy, and if z denote the distance of the centroid of the element dS from the plane, the distance of the centroid of any portion of the spherical surface from the plane is given by the equation

$$\bar{z}=\frac{\int z\,dS}{\int dS}, \tag{1}$$

the integration being extended over the whole portion of the spherical surface considered.

Now if r is the radius of the sphere, the cosine of the angle between the tangent plane to the sphere at the element dS and the plane of xy is $\frac{z}{r}$; therefore

$$d\Sigma = \frac{z}{r}\, dS. \tag{2}$$

Hence $\int z\,dS = r\int d\Sigma = r\Sigma$, Σ denoting the projection of the whole spherical area considered; and making this substitution in (1), we have

$$\bar{z} = r\frac{\Sigma}{S}, \tag{3}$$

where S is the area of that portion of the sphere whose centroid is required.

Equation (1) gives, of course, the distance of the centroid of *any* surface whose element is dS from the plane of xy; and it is clear that if the surface is generated by the motion of a sphere of constant radius whose centre moves along any curve in the plane of xy, the cosine of the angle between the tangent plane at the element dS and the plane of xy will still be $\frac{z}{r}$, since the given surface and the generating sphere have the same tangent plane. Hence equation (2) holds in this case, and therefore also equation (3).

173.] **Centroid of any Surface.** Let dS denote an element of any surface, $d\Sigma$ the projection of this element on the plane of xy, and γ the angle between the plane of xy and the tangent plane to the surface at the element dS. Then, if z is the distance of the centroid of dS from the plane of xy, we have

$$\bar{z} = \frac{\int z\,dS}{\int dS},$$

$$= \frac{\int z\sec\gamma\,.\,d\Sigma}{\int \sec\gamma\,.\,d\Sigma}.$$

It is not unusual to suppose the element dS cut off from the surface in the following manner.

Let m (Fig. 226) be a point in the plane xy whose co-ordinates are x', y'; let mn be drawn parallel to the axis of x and equal to dx'; let mq be parallel to the axis of y and equal to dy'; and complete the rectangle $mnpq$. On the base $mnpq$ describe a prism whose edges, Mm, Nn, Pp, Qq, are parallel to the axis of z. This prism will intercept on the given surface an element, *MNPQ*, which is dS. The rectangular projection, $d\Sigma$, is then *mnpq* whose area is $dx'\,dy'$. Substituting this value in the above equation, we have $\bar{z} = \dfrac{\iint z \sec\gamma\, dx'\,dy'}{\iint \sec\gamma\, dx'\,dy'}$, the integrations being extended over the whole projection of the given surface on the plane xy.

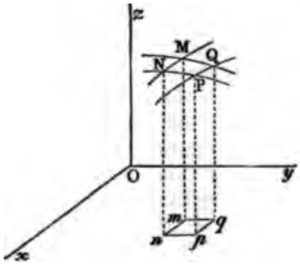

Fig. 226.

It easily follows that *the centroid of the projection (orthogonal or oblique) of any plane area on any plane is the projection of the centroid of the area.*

Take the plane on which the given area is projected as the plane of xy; let ω be the angle between this plane and the plane of the area, and let $\bar{x}$, $\bar{y}$ be co-ordinates of the centroid of the given area. Then

$$\bar{x} = \frac{\int x\,dS}{\int dS} = \frac{\int x \sec\omega\,.\,d\Sigma}{\int \sec\omega\,.\,d\Sigma}$$

$$= \frac{\int x\,d\Sigma}{\int d\Sigma},$$

since ω is the same for all elements. But the co-ordinate of the centroid of the projection is evidently given by this equation. Therefore, &c.; and a similar proof obviously holds for an oblique projection, because at all points of the given area the ratio of dS to $d\Sigma$ is constant.

EXAMPLES.

1. A section of a sphere is made by any two parallel planes; prove that the centroid of the spherical surface included is midway between them.

This is very easily proved either by direct calculation or by the application of the result of last Article. Collignon (*Statique*, p. 295) gives an elegant geometrical demonstration which depends on the fact that if a cylinder is circumscribed to a sphere along any one of its great circles, the portion of the area of the cylinder included between any two planes at right angles to its axis is equal to the portion of the area of the sphere included by these planes. By taking indefinitely close planes it follows that the spherical area may be transferred to the cylinder, and the centroid of any portion of a cylindrical area cut off by planes perpendicular to the axis is evidently midway between these planes.

Cor. The centroid of the surface of a hemisphere is at a distance equal to half the radius from the centre.

The author is indebted to Mr. J. Rendel Harris, of Clare Hall, for the following application of this result to find the centre of mass of a *solid* hemisphere. Suppose AB to be the diameter of the solid hemisphere, O its centre, OE the perpendicular to the plane base at O meeting the surface of the hemisphere at E; draw the tangent plane, CD, at E, the points C and D being on the perpendiculars at B and A to the base AB.

Divide the solid hemisphere into an infinite number of thin concentric hemispherical shells, and replace each shell by a cylinder of the same thickness, having the plane base of this shell for its base, and the radius of this shell for its length. In this way the solid hemisphere will be replaced by the solid which is obtained by cutting out of the cylinder $ABCD$, which surrounds the surface of the given hemisphere, the cone COD. Hence, if r is the radius of the hemisphere, and $\bar{z}$ the distance of the centre of mass of the given solid hemisphere from O,

$$\bar{z} = \frac{\pi r^3 \cdot \dfrac{r}{2} - \frac{1}{3}\pi r^3 \cdot \frac{3}{4} r}{\frac{2}{3}\pi r^3}$$

$$= \tfrac{3}{8} r.$$

2. To find the centroid of a spherical triangle.

Let ABC be any spherical triangle, and O the centre of the sphere. Produce the sides AC and AB until they become quadrants, AE and AD, and draw the arc DE of a great circle.

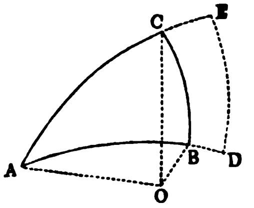

Fig. 227.

We shall find the distance of the centroid from the plane EOD, which is perpendicular to the line OA.

The projection of the area ABC on this plane is evidently the same as the projection of the sector, COB. Now if p_1 is the perpendicular arc from A on the side BC, the angle between the planes COB and EOD is $90^\circ - p_1$; also the area of the sector COB is $\frac{1}{2}ar$, a being the length of the side BC and r the radius of the sphere. Hence if Σ denote the projection of the area of the triangle on the plane EOD,

$$\Sigma = \tfrac{1}{2} ar \sin p_1;$$

and if A, B, C denote the circular measures of the angles of the triangle, and S its area,

$$S = r^2 (A + B + C - \pi).$$

Hence, by (3) of last Article, if x denote the distance of the centroid from the plane,

$$x = \tfrac{1}{2} \cdot \frac{a \sin p_1}{A + B + C - \pi}.$$

It is evident that x is the distance from O of the projection of the centroid on the line OA. Its projections on the lines OB and OC are obtained by writing b and p_2, c and p_3, instead of a and p_1, in this equation.

3. To find the centroid of the surface of a nearly spherical semi-ellipsoid cut off by the plane of the two greater axes.

Let the axes in order of magnitude be a, b, c, and let

$$\frac{a^2-c^2}{a^2}=k^2, \quad \frac{b^2-c^2}{b^2}=k'^2.$$

Now if $dx'dy'$ is the projection on the plane xy (which is the base of the semi-ellipsoid) of an element of surface, dS, we have

$$dS=\frac{c^2dx'dy'}{pz},$$

p being perpendicular from the centre on the tangent plane at the element, and z the distance of the element from the plane of xy. Hence, S denoting the surface of the semi-ellipsoid, we have

$$S.\bar{z}=c^2\iint\frac{dx'dy'}{p}.$$

Again, $$\frac{1}{p^2}=\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z^2}{c^4}=\frac{1}{c^2}\left(1-\frac{k^2x'^2}{a^2}-\frac{k'^2y'^2}{b^2}\right).$$

Therefore, rejecting all powers of k and k' beyond the second

$$S.\bar{z}=c\iint\left(1-\frac{k^2x'^2}{2a^2}-\frac{k'^2y'^2}{2b^2}\right)dx'dy'.$$

Integrating from $x'=-x$ to $x'=x$, the co-ordinates of a point on the circumference of the base being x, y, we have

$$S.\bar{z}=2c\int\left(x-\frac{k^2x^3}{6a^2}-\frac{k'^2xy^2}{2b^2}\right)dy.$$

Expressing x and y in terms of the eccentric angle, and integrating over the entire circumference, we have

$$S.\bar{z}=\pi abc\left(1-\frac{k^2+k'^2}{8}\right)$$

$$=\pi c^3\left\{1+\frac{3}{8}(k^2+k'^2)\right\}.$$

Now (Williamson's *Integral Calculus*),

$$S=\pi a^2b^2c^2\left[\int_0^{\frac{\pi}{2}}\frac{\sin\theta\,d\theta}{(b^2\sin^2\theta+c^2\cos^2\theta)^{\frac{3}{2}}(a^2\sin^2\theta+c^2\cos^2\theta)^{\frac{1}{2}}}\right.$$

$$\left.+\int_0^{\frac{\pi}{2}}\frac{\sin\theta\,d\theta}{(a^2\sin^2\theta+c^2\cos^2\theta)^{\frac{3}{2}}(b^2\sin^2\theta+c^2\cos^2\theta)^{\frac{1}{2}}}\right],$$

which is easily proved to be $2\pi c^2\left\{1+\frac{4}{3}(k^2+k'^2)\right\}$.

Hence finally, $$\bar{z}=\frac{c}{2}\left\{1-\frac{23}{24}(k^2+k'^2)\right\}.$$

4. A parabola revolves round its axis; find the centroid of a portion of the surface between the vertex and a plane perpendicular to the axis at a distance from the vertex equal to $\frac{3}{4}$ of the latus rectum.

Ans. Its distance from the vertex $= \frac{29}{70}$ (latus rectum).

5. Find the centroid of the surface generated by the revolution of a cycloid round its axis.

Ans. It is on the axis at a distance $\frac{2(15\pi-8)}{15(3\pi-4)} \cdot a$ from the vertex, a being the radius of the generating circle.

6. Prove that the centroid of the lateral surface of the frustum of a right cone or pyramid lies in a plane whose distance from the base is $\frac{p+2p'}{3(p+p')} \cdot h$, where p and p' are the perimeters of the base and upper section, and h the height of the frustum.

174.] **Centre of Mass of a Solid of Revolution.** If the curve AB (Fig. 222) revolve round Ox, the rectangular area $PQNM$ will generate a cylindrical volume equal to $\pi . PM^2 . MN$, or $\pi y^2 dx$. Hence if the density of the solid is uniform, we have for the position of its centre of mass (which obviously lies on Ox)

$$\bar{x} = \frac{\int xy^2 dx}{\int y^2 dx},$$

the integrations being extended over the whole of the area $CABD$, of the revolving curve.

If the density varies, the element of mass may require to be taken differently. If the density is a function of x alone, i.e. if it is the same all over the rectangular strip $PQNM$, the volume may be broken up as above, and the element of mass $= \pi\rho y^2 dx$. Hence we shall have, in this case,

$$\bar{x} = \frac{\int \rho x y^2 dx}{\int \rho y^2 dx}.$$

Suppose the density to vary as y alone. Then if we take a small rectangular area, $dx'\, dy'$, at a point whose co-ordinates are x', y', this area will generate an element of volume equal to $2\pi y'\, dx'\, dy'$; therefore the element of mass $= 2\pi\rho y'\, dx'\, dy'$, and we have

$$\bar{x} = \frac{\iint \rho x' y' dx' dy'}{\iint \rho y' dx' dy'}.$$

The integrations are to be performed first from $y' = 0$ to $y' = y$, the ordinate of a point P on the bounding curve; and then from $x' = OC$ to $x' = OD$.

As an example, let the curve *AB* be a quadrant of a circle of which *OA* and *OB* are radii, and let it be required to find the centre of mass of the solid hemisphere generated by the revolution of this quadrant round *OB* (taken as axis of x); firstly, when the density is uniform; secondly, when it is constant over a section perpendicular to *OB* and proportional to the distance of this section from the centre; and thirdly, when it is the same at the same distance from *OB*, and proportional to this distance.

Firstly, we have $\bar{x} = \frac{\int xy^2 dx}{\int y^2 dx}$. Putting $x = r\cos\theta$, $y = r\sin\theta$, where r is the radius of the circle, and integrating between $\theta = 0$ and $\theta = \frac{\pi}{2}$, we have

$$\bar{x} = \frac{3}{8}r. \tag{1}$$

Secondly, since $\rho = \mu x$, we have $\bar{x} = \frac{\int x^2 y^2 dx}{\int xy^2 dx}$, which easily gives

$$\bar{x} = \frac{8}{15}r. \tag{2}$$

Thirdly, $\rho = \mu y'$, therefore

$$\bar{x} = \frac{\int\int x' y'^2 dx' dy'}{\int\int y'^2 dx' dy'} = \frac{\int xy^3 dx}{\int y^3 dx},$$

and the previous substitutions for x and y give

$$\bar{x} = \frac{16}{15\pi}r. \tag{3}$$

In this case the double integration might have been avoided by breaking the area up into rectangles parallel to the axis of x.

The student will do well in such examples as this to check his results as much as possible by a common-sense view of the question. Thus, having proved that the distance of the centre of mass of a homogeneous hemisphere from the centre is $\frac{3}{8}r$, it is clear that when the density of a section is directly proportional to its distance from the centre, the centre of mass of the hemisphere must be at a distance from the centre $> \frac{3}{8}r$, since the matter is most dense in the space remote from the centre; while in the third case above, since the ordinates of the portion of the curve near *A* are greater than those of the portion near *B*, and since the density increases with the ordinate, it is evident that the centre of mass must be nearer to the centre than in the homogeneous hemisphere.

The most advantageous method of breaking up a mass of varying density into elements depends entirely on the law of variation of the density, and while all these methods are embraced in the rule of Art. 165, it would be impossible to give formulae suited to all cases.

Laplace, by assuming the change of the pressure from stratum to stratum of the earth to be proportional to the change in the square of the density, proves that if the strata of uniform density are spherical, the density of a stratum of radius x is given by the equation

$$\rho = \frac{a\rho_0}{\mu} \cdot \frac{\sin\frac{\mu x}{a}}{x},$$

a being the radius of the earth, ρ_0 the density at the centre, and μ a constant number.

Let it be required to find the centre of mass of a hemisphere whose density follows this law.

Here the element of mass of uniform density is the stratum included between the hemispheres of radii x and $x+dx$. Hence

$$\begin{aligned} dm &= 2\pi\rho x^2 dx \\ &= 2\,\pi a\rho_0 \frac{x}{\mu} \sin\frac{\mu x}{a}\,dx. \end{aligned}$$

Also the distance of the centre of mass of this stratum from the centre is $\frac{x}{2}$ (Example 1, p. 325). Hence, the axis of x being the diameter perpendicular to the base of the hemisphere, the distance of the centre of mass from the centre is given by the equation

$$\begin{aligned} x &= \tfrac{1}{2}\frac{\int_0^a x^2 \sin\frac{\mu x}{a}\,dx}{\int_0^a x \sin\frac{\mu x}{a}\,dx} \\ &= a \cdot \frac{(2-\mu^2)\cos\mu + 2\mu\sin\mu - 2}{2\mu(\sin\mu - \mu\cos\mu)}, \end{aligned}$$

as will be easily found. When $\mu = 0$ the hemisphere is of uniform density, and the student will see that this value of $\bar{x}$ becomes $\frac{3}{8}a$, in accordance with our previous result.

EXAMPLES.

1. Find the centre of mass of a hemisphere in which the density is proportional to the n^{th} power of the distance from the centre.

Ans. It is at a distance $= \frac{n+3}{n+4} \cdot \frac{a}{2}$ from the centre, a being the radius of the hemisphere.

2. Find the centre of mass of a portion of a paraboloid of revolution cut off by a plane perpendicular to its axis.

Ans. If h is the distance of the plane of section from the vertex, $\bar{x} = \frac{3}{8} h$.

3. Find the centre of mass of a semi-ellipsoid of revolution round the minor axis, the density at any point being proportional to its distance from the base which is the plane perpendicular to the axis of revolution.

Ans. $\bar{y} = \frac{8}{15} b$, where b is the semi-minor axis.

4. An ellipsoid of revolution round the minor axis is cut by a plane passing through this axis; find the centre of mass of the portion included between one semi-ellipsoid thus cut off and the concentric hemisphere whose diameter is the minor axis.

Ans. If a and b are the axes major and minor of the generating ellipse, the required centre of mass is on the major axis at a distance equal to $\frac{3}{8} \cdot \frac{a^2+ab+b^2}{a+b}$ from the centre.

Verify this result in two obvious cases.

175.] **Centre of Mass of any Solid.** In the solid take any point, P, whose co-ordinates are x, y, z, and also a close point, Q, whose co-ordinates are $x+dx$, $y+dy$, $z+dz$. Then evidently the volume of the parallelopiped whose diagonal is PQ and whose edges are parallel to the axes of co-ordinates is $dx\,dy\,dz$; and if ρ is the density of the body at P the element of mass at P is $\rho\,dx\,dy\,dz$.

Hence the co-ordinates of the centre of mass of the solid are given by the equations

$$\bar{x} = \frac{\iiint \rho x\,dx\,dy\,dz}{\iiint \rho\,dx\,dy\,dz}, \quad \bar{y} = \frac{\iiint \rho y\,dx\,dy\,dz}{\iiint \rho\,dx\,dy\,dz}, \quad \bar{z} = \frac{\iiint \rho z\,dx\,dy\,dz}{\iiint \rho\,dx\,dy\,dz},$$

the integrations being extended over the whole solid.

It may not be necessary to take infinitesimal elements of volume of the third order. From what has preceded, the student will have learned that the best mode of breaking up the given mass into elements depends entirely on the law of density which prevails.

In many cases the symmetry of the solid enables us to simplify the problem by choosing elements of volume which are infinitesimals of the *first* order only.

The various elements of volume which it may be necessary to take are exemplified in the following problems.

Find the centre of mass of the eighth part of an ellipsoid, ABC (Fig. 228), included between its three principal planes—

(1) When the density at any point is simply a function of its distance from the principal plane *BC* (plane of yz).

(2) When the density at any point is a function of its distances from the two principal planes *AC* and *BC* (planes of xz and yz).

(3) When the density at any point is a function of its distances from the three principal planes.

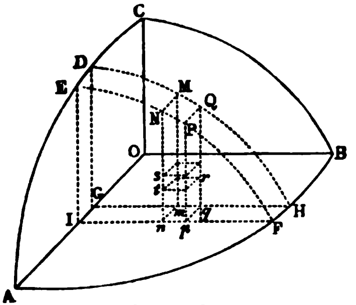

Fig. 228.

In the first case, the density will be constant over a section *DH* perpendicular to *OA*. Hence, taking two such sections, *DH* and *EF*, at a distance dx from each other, the density of the solid between them may be considered uniform, and this portion of the solid may be taken as the element of mass.

In the second case, the density will be constant throughout a portion of the body in which x and y are constant; that is, along a perpendicular to the plane *AB*; and the element of mass may be taken as the prism $NQnq$, the area of whose base is $dx\,dy$, and which intersects the bounding surface in the area $NMQP$.

In the third case, the density is not the same at any two points, and the element of mass must be taken as a small rectangular prism, str, whose volume is $dx\,dy\,dz$.

EXAMPLES.

1. In the problem just discussed, find the centre of mass when the density at any point is proportional to its distance from the plane *BC*.

Here $\rho = \mu x$; also, the equation of the ellipsoid being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

the ellipse *DH* satisfies the equation

$$\frac{y^2}{b^2\left(1 - \frac{x^2}{a^2}\right)} + \frac{z^2}{c^2\left(1 - \frac{x^2}{a^2}\right)} = 1,$$

which shows that the axes GH and GD are

$$b\sqrt{1-\frac{x^2}{a^2}} \text{ and } c\sqrt{1-\frac{x^2}{a^2}},$$

respectively. Hence, IG being $= dx$, the element of mass is

$$\pi\mu bcx\left(1-\frac{x^2}{a^2}\right)dx;$$

and since the centre of mass of this element is ultimately a point whose co-ordinates are

$$x, \frac{4b}{3\pi}\sqrt{1-\frac{x^2}{a^2}}, \text{ and } \frac{4c}{3\pi}\sqrt{1-\frac{x^2}{a^2}}$$

(see Ex. 3, p. 312), we have

$$\bar{x} = \frac{\int_0^a x^2\left(1-\frac{x^2}{a^2}\right)dx}{\int_0^a x\left(1-\frac{x^2}{a^2}\right)dx} = \frac{8}{15}a;$$

and

$$\bar{y} = \frac{4b}{3\pi}\cdot\frac{\int_0^a x\left(1-\frac{x^2}{a^2}\right)^{\frac{3}{2}}dx}{\int_0^a x\left(1-\frac{x^2}{a^2}\right)dx} = \frac{16b}{15\pi};$$

the value of $\bar{z}$ being, of course, $\frac{16c}{15\pi}$.

2. If the density at any point of the ellipsoid is μxy, find the centre of mass.

Taking a prismatic element of volume $NQnq$, the element of mass is

$$\mu xyz\,dx\,dy,$$

z being the height, Mm, of the prism.

The co-ordinates of M being x, y, z, those of the centre of mass of this prism are evidently x, y, $\frac{z}{2}$. Hence

$$\bar{x} = \frac{\iint x^2yz\,dx\,dy}{\iint xyz\,dx\,dy}, \quad \bar{y} = \frac{\iint xy^2z\,dx\,dy}{\iint xyz\,dx\,dy}, \quad \bar{z} = \frac{1}{2}\frac{\iint xyz^2\,dx\,dy}{\iint xyz\,dx\,dy}.$$

The integrations may be performed, first with regard to y, from $y = 0$ to $y = GH$; and then with regard to x, from $x = 0$ to $x = OA$.

$$\text{Now, } \iint xyz\,dx\,dy = c\iint xy\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}}dx\,dy;$$

and, integrating first with regard to y, we have

$$\int_0^{GH} y\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}}dx = \frac{b^2}{3}\left(1-\frac{x^2}{a^2}\right)^{\frac{3}{2}},$$

since from the equation of the ellipse AB, the value GH of y makes $1-\frac{x^2}{a^2}-\frac{y^2}{b^2}$ vanish. Hence

$$\int\int xyz\,dxdy=\frac{b^2c}{3}\int_0^a x\left(1-\frac{x^2}{a^2}\right)^{\frac{3}{2}}dx=\frac{a^2b^2c}{15}.$$

In the same way,

$$\int\int x^2yz\,dxdy=\frac{b^2c}{3}\int_0^a x^2\left(1-\frac{x^2}{a^2}\right)^{\frac{3}{2}}dx,$$

which, by putting $x=a\cos\phi$, is easily seen to be $\frac{\pi a^3b^2c}{96}$. Hence $\bar{x}=\frac{5\pi}{32}\cdot a$, and $\bar{y}=\frac{5\pi}{32}\cdot b$; and it is easily found that $\bar{z}=\frac{5}{8}c$.

3. If the density at any point in the solid is proportional to the product of the co-ordinates of the point, find the centre of mass.

Here, at any point, we have $\rho=\mu.xyz$, and the element of mass being $\mu xyzdxdydz$, we have

$$x=\frac{\int\int\int x^2yz\,dxdydz}{\int\int\int xyz\,dxdydz},$$

with similar values of $\bar{y}$ and $\bar{z}$. If we first integrate from $z=0$ to $z=mM$ (Fig. 228), we shall have the contribution of the prism $NQnq$ to the summation. Integrating, then, with respect to z, considering x and y constant, we have

$$\int\int\int x^2yz\,dxdydz=\tfrac{1}{2}\int\int x^2y(mM)^2.\,dxdy$$
$$=\frac{c^2}{2}\int\int x^2y\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dxdy,$$

since M is a point on the bounding surface of the ellipsoid. Let this latter integration be first performed with respect to y, considering x constant, from $y=0$ to $y=GH$, and we shall then have the contribution of the mass contained between the sections DH and EF.

Now $$\int_0^{GH}y\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dy=\frac{b^2}{4}\left(1-\frac{x^2}{a^2}\right)^2.$$

Hence $$\int\int\int x^2yz\,dxdydz=\frac{b^2c^2}{8}\int_0^a x^2\left(1-\frac{x^2}{a^2}\right)^2dx=\frac{a^3b^2c^2}{105},$$

as easily appears by putting $x=a\cos\phi$.

It will be found without difficulty that $\int\int\int xyz\,dxdydz=\frac{a^2b^2c^2}{48}$.

Hence $$\bar{x}=\frac{16}{35}a,\quad \bar{y}=\frac{16}{35}b,\ \text{and}\ \bar{z}=\frac{16}{35}c.$$

4. Find the centre of mass of the portion of the elliptic paraboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2\frac{z}{c}$ included between the planes xz and yz and a plane perpendicular to the axis of z at a distance h from the vertex.

Ans. $\bar{x} = \frac{16a}{15\pi}\sqrt{\frac{2h}{c}}, \quad \bar{y} = \frac{16b}{15\pi}\sqrt{\frac{2h}{c}}, \quad \bar{z} = \frac{2}{3}h.$

5. At each point, M, in the semi-axis major of an ellipse, is drawn a line perpendicular to the plane of the ellipse, its length being proportional to the distance of M from the centre; the extremity of this perpendicular is joined to the point P on one quadrant of the ellipse such that PM is perpendicular to the axis major. Find the centroid of the volume thus generated.

Ans. If at any distance, x, from the centre the perpendicular to the plane of the ellipse is kx, and if the axes of x, y, and z are the axes of the ellipse and a perpendicular to them, we have

$$\bar{x} = \frac{3\pi a}{16}, \quad \bar{y} = \frac{b}{4}, \quad \bar{z} = \frac{\pi k a}{16}.$$

6. Through a diameter of the base of a right cone are drawn two planes cutting the cone in parabolas; find the centroid of the volume of the cone included between these planes and the vertex.

Ans. It is on the axis at a distance from the vertex equal to $\frac{3}{5}$ of height of cone.

7. A plane cuts off a constant volume from an ellipsoid; find the locus of the centroid of the portion cut off.

Ans. An ellipsoid similar to the given one, and similarly placed (see Example 2, p. 312, the theorem of which is equally applicable to surfaces).

176.] **Polar Elements of Mass.** Let Fig. 229 represent the portion of the volume of a solid included between its bounding surface and three rectangular co-ordinate planes. Then the solid may be broken up into elements in the following manner:—

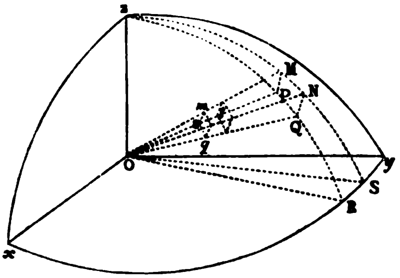

Fig. 229.

(1) Through the axis of z draw two close planes cutting the bounding surface in curves zR and zS (called *meridians*); and let the angles ROx and SOx be denoted by ϕ and $\phi + d\phi$, respectively.

(2) Round the axis of z describe two right cones with the semi-vertical angles zOP and zOQ, equal to θ and $\theta + d\theta$, respectively.

since from the equation of the ellipse AB, the value GH of y makes $1-\frac{x^2}{a^2}-\frac{y^2}{b^2}$ vanish. Hence

$$\int\int xyz\,dx\,dy=\frac{b^2c}{3}\int_0^a x\left(1-\frac{x^2}{a^2}\right)^{\frac{3}{2}}dx=\frac{a^2b^2c}{15}.$$

In the same way,

$$\int\int x^2yz\,dx\,dy=\frac{b^2c}{3}\int_0^a x^2\left(1-\frac{x^2}{a^2}\right)^{\frac{3}{2}}dx,$$

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with similar values of $\bar{y}$ and $\bar{z}$. If we first integrate from $z=0$ to $z=mM$ (Fig. 228), we shall have the contribution of the prism $NQnq$ to the summation. Integrating, then, with respect to z, considering x and y constant, we have

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$$=\frac{c^2}{2}\int\int x^2y\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dx\,dy,$$

since M is a point on the bounding surface of the ellipsoid. Let this latter integration be first performed with respect to y, considering x constant, from $y=0$ to $y=GH$, and we shall then have the contribution of the mass contained between the sections DH and EF.

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Hence $$\int\int\int x^2yz\,dx\,dy\,dz=\frac{b^2c^2}{8}\int_0^a x^2\left(1-\frac{x^2}{a^2}\right)^2dx=\frac{a^3b^2c^2}{105},$$

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Ans. $\bar{x} = \frac{16a}{15\pi}\sqrt{\frac{2h}{c}}$, $\bar{y} = \frac{16b}{15\pi}\sqrt{\frac{2h}{c}}$, $\bar{z} = \frac{2}{3}h$.

5. At each point, M, in the semi-axis major of an ellipse, is drawn a line perpendicular to the plane of the ellipse, its length being proportional to the distance of M from the centre; the extremity of this perpendicular is joined to the point P on one quadrant of the ellipse such that PM is perpendicular to the axis major. Find the centroid of the volume thus generated.

Ans. If at any distance, x, from the centre the perpendicular to the plane of the ellipse is kx, and if the axes of x, y, and z are the axes of the ellipse and a perpendicular to them, we have

$$\bar{x} = \frac{3\pi a}{16}, \quad \bar{y} = \frac{b}{4}, \quad \bar{z} = \frac{\pi ka}{16}.$$

6. Through a diameter of the base of a right cone are drawn two planes cutting the cone in parabolas; find the centroid of the volume of the cone included between these planes and the vertex.

Ans. It is on the axis at a distance from the vertex equal to $\frac{3}{5}$ of height of cone.

7. A plane cuts off a constant volume from an ellipsoid; find the locus of the centroid of the portion cut off.

Ans. An ellipsoid similar to the given one, and similarly placed (see Example 2, p. 312, the theorem of which is equally applicable to surfaces).

176.] **Polar Elements of Mass.** Let Fig. 229 represent the portion of the volume of a solid included between its bounding surface and three rectangular co-ordinate planes. Then the solid may be broken up into elements in the following manner:—

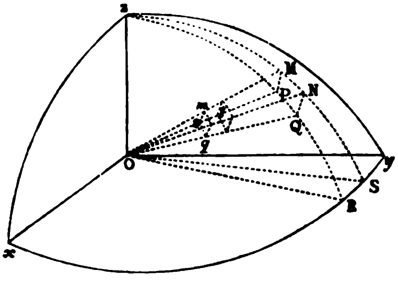

Fig. 229.

(1) Through the axis of z draw two close planes cutting the bounding surface in curves zR and zS (called *meridians*); and let the angles ROx and SOx be denoted by ϕ and $\phi + d\phi$, respectively.

(2) Round the axis of z describe two right cones with the semi-vertical angles zOP and zOQ, equal to θ and $\theta + d\theta$, respectively.

(3) With O as centre, describe two close spheres whose radii, Os and Ot, are equal to r and $r+dr$, respectively.

These planes, cones, and spheres will then determine the small rectangular parallelopiped $mstq$, whose volume $= ms \times sq \times st$.

Now, perpendiculars from m and s on Oz will each be equal to $Os \,.\, \sin zOs$, or $r \sin\theta$, and they will include an angle equal to ROS, or $d\phi$; therefore $ms = r \sin\theta \, d\phi$. Also,

$$sq = Os \,.\, \sin sOq = r d\theta \text{; and } st = dr.$$

Therefore the volume of the elementary parallelopiped $= r^2 \sin\theta \, dr \, d\theta \, d\phi$; and if ρ is the density of the solid at s, the element of mass is $\rho r^2 \sin\theta \, dr \, d\theta \, d\phi$.

Again, the co-ordinates of the centre of mass of this element are ultimately the same as those of s; therefore they are

$$r \sin\theta \cos\phi, \quad r \sin\theta \sin\phi, \quad \text{and} \quad r\cos\theta;$$

and for the centre of mass of any finite portion of the solid we have

$$\bar{x} = \frac{\iiint \rho r^3 \sin^2\theta \cos\phi \, dr \, d\theta \, d\phi}{\iiint \rho r^2 \sin\theta \, dr \, d\theta \, d\phi},$$

$$\bar{y} = \frac{\iiint \rho r^3 \sin^2\theta \sin\phi \, dr \, d\theta \, d\phi}{\iiint \rho r^2 \sin\theta \, dr \, d\theta \, d\phi},$$

$$\bar{z} = \frac{\iiint \rho r^3 \sin\theta \cos\theta \, dr \, d\theta \, d\phi}{\iiint \rho r^2 \sin\theta \, dr \, d\theta \, d\phi},$$

the limits of integration being determined by the figure of the portion of the solid considered.

The angles θ and ϕ are sometimes called the *colatitude* and *longitude*, respectively.

EXAMPLES.

1. Find the centre of mass of a portion of a solid sphere contained in a right cone whose vertex is the centre of the sphere, the density of the solid varying as the n^{th} power of the distance from the centre.

Take the axis of the cone as that of z, and any plane through it as that from which longitude is measured. Then it is clear that $\bar{x} = \bar{y} = 0$, and we have

$$\bar{z} = \frac{\iiint r^{n+3} \sin\theta \cos\theta \, dr \, d\theta \, d\phi}{\iiint r^{n+2} \sin\theta \, dr \, d\theta \, d\phi}.$$

Performing the integration first with respect to r, considering θ and ϕ constant, from $r = 0$ to $r = a$, the radius of the sphere, we have

$$\bar{z} = \frac{n+3}{n+4} a \frac{\iint \sin\theta \cos\theta \, d\theta \, d\phi}{\iint \sin\theta \, d\theta \, d\phi}.$$

Performing the integration now with respect to ϕ, the longitude, which runs from 0 to 2π, we have

$$\bar{z} = \frac{n+3}{n+4}\, a\, \frac{\int \sin\theta \cos\theta\, d\theta}{\int \sin\theta\, d\theta}\,.$$

If $a =$ the semi-vertical angle of the cone, the limits of θ are 0 and a.

Therefore $$\bar{z} = \frac{n+3}{n+4} \cdot \frac{a}{2}(1+\cos a).$$

2. Find the centre of mass of a prism whose base is a given spherical triangle and whose vertex is the centre of the sphere on which the triangle is described.

Let O (Fig. 227) be the centre of the sphere, and take OC as axis of z. From C draw the perpendicular p_3 to the base AB, and let R be the radius of the sphere.

The value of $\bar{z}$ given as a triple integral may be modified in the present case.

Let dS denote any small element of area at any point P; then the volume of a cone whose base is this element and vertex the centre of the sphere is $\frac{1}{3} R\,dS$, and the distance of its centre of mass from the plane of xy is (Art. 163) $\frac{3}{4} R\cos\theta$. Hence

$$\bar{z} = \frac{3}{4} R \frac{\int \cos\theta\, dS}{\int dS}\,.$$

Now $\cos\theta\,.\,dS$ is the projection of the element dS on the plane of xy; therefore the numerator is the projection of the whole area ABC on this plane, which, as in Example 2, p. 326, is $\frac{1}{2} cR \sin p_3$. Hence,

$$\bar{z} = \frac{3}{8} \frac{c \sin p_3}{A+B+C-\pi}\,.$$

3. A cardioid revolves round its axis; find the centre of mass of the solid generated.

Ans. It is at a distance from the cusp equal to $\frac{2}{5}$ of the axis.

177.] **Theorems of Pappus.** *If a plane area revolve through any angle round a line in its plane, the volume generated is equal to the area of the revolving figure multiplied by the length of the path described by its centroid.*

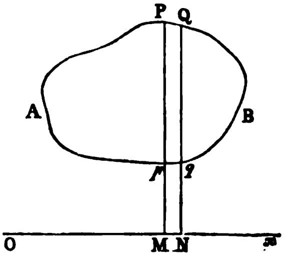

Fig. 230.

Let AB (Fig. 230) be the revolving figure, and Ox the line about which it revolves. Let the area be broken up into an indefinitely great number of rectangular strips, such as $PQqp$, by lines perpendicular to Ox. Then the volume generated by this strip in revolving through an angle ω is evidently equal to

$$\frac{\omega}{2\pi}.\pi(PM^2-Mp^2).MN,$$

or

$$\tfrac{1}{2}\omega(y_2^2-y_1^2)\,dx,$$

denoting PM, pM, and MN by y_2, y_1, and dx. Hence if V denote the whole volume generated,

$$V=\tfrac{1}{2}\omega\int(y_2^2-y_1^2)\,dx.$$

Now the distance of the centroid of the strip from Ox is $\frac{y_2+y_1}{2}$; and the area of the strip is $(y_2-y_1)\,dx$. Hence, denoting these quantities by y and dA respectively,

$$V=\omega\int y\,dA$$
$$=\omega A.\bar{y},$$

A denoting the whole revolving area and $\bar{y}$ the ordinate of its centroid. Now in revolving through the angle ω, the centroid of the area describes a circular arc whose length is $\omega\bar{y}$. Hence the theorem is proved.

If the axis Ox intersects the revolving figure, the theorem still applies with the convention that the volumes generated by the portions of the figure at opposite sides of Ox are affected with opposite signs.

Again, *if the arc of any plane curve revolve through any angle round a line in its plane, the area of the surface generated is equal to the length of the revolving arc multiplied by the length of the path described by its centroid.*

For, the surface generated is

$$\omega\int y\,ds, \quad \text{or} \quad \omega L.\bar{y},$$

L being the whole length of the revolving arc and $\bar{y}$ the ordinate of its centroid. As before, $\omega\bar{y}$ is the length of the circular arc described by the centroid of the revolving arc, and the theorem is evidently proved.

If the revolving arc intersects the line Ox, the theorem is true, with the previous convention of signs.

178.] **Extension of the Theorems of Pappus.** The previous theorems can be easily extended to the case in which the plane of the revolving figure, instead of revolving round a fixed line, *rolls without sliding on any developable surface*, and the first theorem will then become—

If the plane of any plane area rolls without sliding on a developable surface, the volume generated by the area in moving from one position to another will be equal to the area of the revolving figure multiplied by the length of the path described by its centroid.

A similar enunciation gives the second theorem.

These propositions are evidently true, because in an indefinitely small motion the figure is revolving round a generating line of the developable, and for such a small motion the theorem of Pappus gives the volume generated equal to the area × small space described by its centroid. Taking the sum of all such elements of volume from one position of the figure to another, we have the theorem of this Article.

It is clear also that the theorems hold in the case of a plane area which moves in such a manner as to be always normal to the path described by its centroid. For the area may at any instant be considered as revolving round the line of intersection of two consecutive normal planes of the curve which the centroid describes, and the theorems are then directly applicable.

179.] **Volume of a Truncated Cylinder or Prism.** Let A and B denote the sections of a cylinder or prism made by any two planes. Through any line L passing through the centroid, G, of B draw any plane, B', inclined at an angle to B. Then G is the centroid of the section B', since this section is the projection of B made by lines parallel to the generators of the cylinder or edges of the prism, and since (Art. 173) the centroid of the projection of any plane surface is the projection of its centroid. Also, the volume between the sections B and B' on one side of the line $L =$ the volume between them on the other side, as we see by breaking up the area B into an infinite number of infinitely small elements of area, and constructing slender prisms on each one, dS, of these elements, the prisms being terminated by B'. If y is the perpendicular on L from the middle of dS, and $\gamma =$ angle between the planes B, B', the volume of the corresponding prism $= \tan\gamma \,.\, y\,dS$; therefore the total volume between the planes B, B' is $\tan\gamma\int y\,dS$, which $= 0$, because L passes through the centroid of the area B. In other words, the volume of the prism or cylinder contained between the sections A and B is equal to that contained between the sections A and B'. Allowing B' to revolve again about L through any angle, the same reasoning applies, and we see, finally, that for the sections A and B

may be substituted *any* two passing through their respective centroids, and the included volume will be unaltered. Let two parallel sections, each perpendicular to the axis of the prism or cylinder, be substituted, and the included volume will be $\Omega \,.\, h$, where Ω is the area of either normal section and h the distance between them.

180.] **Equilibrium of a Heavy Body on a Horizontal Plane.** When an indeformable body rests on a horizontal plane, the contact taking place at several points, either continuous or not, it is kept in equilibrium by two forces—namely, its own weight and the reaction of the plane. The condition necessary and sufficient for the equilibrium of such a body is that these two forces must be equal and opposite. Now this will be impossible unless the points of contact of the body with the plane can be so connected by right lines as to form a polygon within the area of which the vertical through the centre of gravity of the body intersects the plane. For, whether the plane be rough or smooth, resolve all the reactions at the points of contact vertically. Then it is evident that the resultant of the system of parallel vertical forces at the points of contact must necessarily fall within some polygon whose vertices are these points; therefore, &c.

The student must be careful to observe that this condition, though necessary in the case of a deformable system, is not sufficient (see Article 86). Thus, in Example 16, p. 253, it is not true that the deformable system of two bars, AB and BC, will rest in any position in which their common centre of gravity falls between the props.

EXAMPLES.

1. To find the volume and surface of a tore.

(A tore is a surface generated by the revolution of a circle round a line in its plane.)

Let r be the radius of the circle, and c the distance of its centre from the axis of revolution. Then the volume of the tore is evidently $\pi r^2 \times 2\pi c$, or $2\pi^2 cr^2$; and the surface is $2\pi r \times 2\pi c$, or $4\pi^2 cr$.

2. A triangle revolves round a line in its plane; find the volume generated.

Ans. If the distances of the vertices from the lines are x_1, x_2, x_3, and A the area of the triangle, the volume $= \frac{2\pi A}{3}(x_1 + x_2 + x_3)$.

3. From the Theorems of Pappus deduce the volume and surface of a frustum of a right cone.

(Consider a trapezium one side of which is perpendicular to the two parallel sides.)

4. A pack of cards is laid on a table; each projects in the direction of the length of the pack beyond the one below it; if each projects as far as possible, prove that the distances between the extremities of the successive cards will form a harmonic progression. (Walton, p. 183.)

5. A rectangular column is formed by placing a number of smooth cubical blocks one above another, the base of the column resting on a horizontal plane; all the blocks above the lowest are then turned in the same direction about an edge of the column, first the highest, then the highest two, and so on, in each case as far as is consistent with equilibrium. Prove that the sum of the sines of the inclinations of a diagonal of the base of any block to the like diagonals of the bases of all the blocks above it is equal to the sum of the cosines. (Walton, *ibid.*)

CHAPTER XII.

EQUILIBRIUM OF FLEXIBLE STRINGS.

181.] **Perfectly Flexible String.** A string is said to be *perfectly flexible* when at every point in its length it can be bent round all lines passing through the point perpendicularly to the tangent line without the expenditure of work.

From this definition it follows that the internal force, or mutual action between the particles at each side of any normal section of such a string, has no component in the plane of the section; this force must, therefore, be entirely normal to the section; or, in other words, *the internal force in a perfectly flexible string is at every point directed along the tangent line to the string.*

This internal force we have called the *tension* of the string, and, like all internal forces in a system, it is a mutual action between parts of the system. This has been sufficiently explained already (p. 31). In the sequel we shall use the term *flexible string* as equivalent to perfectly flexible string.

182.] **Imperfectly Flexible String.** No effort is required to bend a perfectly flexible string at any point; but if we attempt to bend an imperfectly flexible string, or a wire, we encounter a certain amount of resistance according to the degree of inflexibility or rigidity of the string or wire. If we consider the nature of the mutual forces existing between the particles on each side of a normal section of such a body, we shall find that these forces are not necessarily reducible to a single resultant at all. In the general case of a wire bent and twisted by the action of any external forces, these internal actions on the particles at one side of a section may, of course, be reduced to a single resultant force and a single couple; and the resultant force may be applied at any point in the section, the couple varying according to the point chosen. All this will be evident from the general reduction of a system of forces in Chapter XIV.

183.] **Three Methods of Investigation.** There are three methods by which the equilibrium of a string or wire may be treated—namely,

1°. We may isolate an infinitesimal element of the body, supplying to it at each extremity the action exercised by the neighbouring portions which are imagined to be removed. This is the principle of separate equilibrium. (Art. 34.)

2°. We may apply the general condition that for any system of imagined small displacements—involving, of course, in general, slight extensions of the elements of the string—the whole work of the system of forces, internal as well as external, is zero (see p. 121).

3°. We may consider the equilibrium of any *finite* portion of the body, treating it, *when the figure of equilibrium has been assumed* (see p. 17), *as a rigid body.*

(See Thomson and Tait, *Nat. Phil.*)

We begin by considering the equilibrium of a perfectly flexible string which suffers no elongation under the action of the forces which will keep it in equilibrium. Such a body is called a *flexible inextensible string*, and it is scarcely necessary to add that it exists only in the abstractions of Rational Statics.

Section I.

Flexible Inextensible Strings.

184.] **The Common Catenary.** Before deducing the general equations of equilibrium of a string acted upon by any system of coplanar forces, we shall discuss the case of a string hanging under the action of gravity. And firstly, let us suppose the string to be inextensible and of uniform cross-section and density. Let its extremities be fixed at two points *B* and *C* (Fig. 231) which, so far as the general properties of the curve formed by the string are concerned, need not be in the same horizontal line,

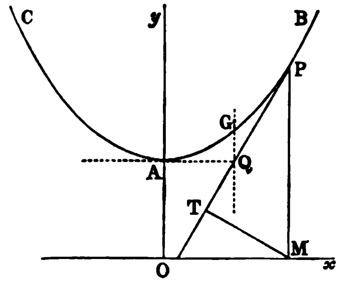

Fig. 231.

nor need there be a horizontal tangent to the curve at some point of the string between them, although the figure represents a case in which there is such a tangent (at A). The first obvious property of the curve of equilibrium is one that has already appeared in the case of a funicular polygon (p. 41) from whose vertices any masses, equal or unequal, are suspended—viz., *the horizontal component of the tension is constant at all points on the curve.* For, let P be any point on the curve, A the lowest point, the tensions at these points being T and τ, respectively, and consider the separate equilibrium of the portion AP of the string. Since gravity produces a vertical force acting through the centre of gravity, G, of the portion AP, if we resolve horizontally we have simply

$$T \cos \theta = \tau, \tag{1}$$

where θ is the angle which the tangent QP at P makes with the horizon. Moreover the string must take such a shape that the vertical, GQ, through the centre of gravity of the arc AP passes through the point, Q, of intersection of the tangents at A and P. Indeed, it is obvious that the tangents at *any* two points, P, P', of the curve must intersect on the vertical through the centre of gravity of the portion PP' of the string; for, suppose P' to be any point on the curve between A and P, and consider the separate equilibrium of the portion $P'P$. Three forces keep it in equilibrium, viz., the tensions along the tangents at P' and P, and its weight. These forces must meet in a point (Art. 105), therefore, &c.

Again, for the equilibrium of the portion AP resolve vertically. If w is the weight of the string per unit length, and if the length AP is s, we have

$$T \sin \theta = w \,.\, s. \tag{2}$$

Take the horizontal and vertical lines through A as axes of x and y, respectively. Then if (x, y) are the co-ordinates of P, we have $\cos \theta = \frac{dx}{ds}$, $\sin \theta = \frac{dy}{ds}$; also (1) and (2) give

$$\tan \theta = \frac{w \,.\, s}{\tau}, \tag{3}$$

i.e., the tangent of the inclination of the curve at any point, P, is proportional to the length of the arc, AP, measured from the lowest point. The tension τ may be regarded as equal to the weight of a certain length, c, of the string. Put, then, $\tau = w \cdot c$, and we have

$$\tan\theta = \frac{s}{c}. \qquad (4)$$

This shows that if along the tangent at P we measure a length PT equal to the length, s, of AP, and at T erect the perpendicular TM to meet the vertical PM through P in M, the length TM is constant whatever point, P, is chosen and is equal to c. Hence

$$PM = \sqrt{c^2+s^2}. \qquad (5)$$

The equation of the curve is easily found from (4). Thus, since $\tan\theta = \frac{dy}{dx}$, we can obtain the relation between y and s:

$$dy = \frac{s}{c}\,dx, \qquad (6)$$

$$= \frac{s}{c}\sqrt{ds^2-dy^2},$$

$$\sqrt{c^2+s^2}\,.\,dy = s\,ds,$$

which gives by integration

$$y = \sqrt{c^2+s^2}+C, \qquad (7)$$

where C is some constant. For the point A, we have, with the chosen axes, $y = 0$, $s = 0$, $\therefore C = -c$, so that (7) becomes

$$y+c = \sqrt{c^2+s^2}$$

$$= PM. \qquad (8)$$

Now instead of drawing the axis of x at A, draw it at a distance c below A. Let $AO = c$, and let Ox be the new axis of x. This point O we shall call *the origin* of the curve, and the line Ox its *directrix*. Then the new y of P is $c+$ the previous y. Hence the point M which we previously determined as the intersection of a perpendicular to the tangent PT at T (where $PT = s$) lies on the new axis of x whatever be the position of P on the curve, and therefore

$$y = PM$$

$$= \sqrt{c^2+s^2}. \qquad (9)$$

We can find the relation between x and s from (6) by putting $dy = \sqrt{ds^2-dx^2}$. This gives

$$dx = \frac{c\,ds}{\sqrt{c^2+s^2}}, \qquad (10)$$

which gives by integration

$$\frac{x}{c} = \log_e(s+\sqrt{c^2+s^2})+C,$$

where C is some constant, and e denotes the Napierian base of logarithms. Now at A we have $x = 0$, $s = 0$, $\therefore\ C = -\log_e c$, so that

$$\frac{x}{c} = \log_e \frac{s + \sqrt{c^2 + s^2}}{c}, \tag{11}$$

$$\therefore\quad e^{\frac{x}{c}} = \frac{s + \sqrt{c^2 + s^2}}{c}, \tag{12}$$

$$\therefore\quad s = \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right). \tag{13}$$

In the notation of hyperbolic functions

$$\tfrac{1}{2}\left(e^{\theta} + e^{-\theta}\right) \equiv \cosh\theta\,; \quad \tfrac{1}{2}\left(e^{\theta} - e^{-\theta}\right) \equiv \sinh\theta, \tag{14}$$

so that for the curve of equilibrium of a uniform string under the action of gravity, we have

$$s = c \sinh\frac{x}{c}. \tag{15}$$

Again, since $y = \sqrt{s^2 - c^2} = c\sqrt{\sinh^2\frac{x}{c} - 1}$, we have

$$y = c\cosh\frac{x}{c}$$

$$= \frac{c}{2}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right), \tag{16}$$

which is the simplest equation of the curve in rectangular co-ordinates.

The curve of equilibrium of a uniform flexible inextensible string under the action of gravity, and suspended from two points, is called the *Common Catenary*. A uniform chain with fine links, suspended from two points, may also be considered as forming the Common Catenary.

If one end of a string is fixed at B, and the other end at any point such as P, we may determine the curve $BPAC$ of which the given portion, BP, of string is part, and we may imagine the point P released and the string continued through P to any other point on the curve and there fixed. If this were done, the portion BP would be in precisely the same state as that in which it is actually when the string has its ends fixed at B and P. It is well to set down briefly a few of the more prominent geometrical properties of the catenary.

Since at all points P the length MT is equal to c, *the length of the perpendicular on the tangent let fall from the foot of the ordinate is constant.*

Hence, given the catenary, to construct its origin and horizontal axis—

On the tangent at any point, P, measure off a length, PT, equal to the arc AP; at T erect a perpendicular TM to the tangent meeting the ordinate of P in M; then the horizontal line through M is the directrix of the curve.

In making a proper figure this rule will be found of great use.

The involute of the catenary which starts from the lowest point is the Tractory.

To get a point on this involute we measure on the tangent, PT, at any point, P, a length equal to the arc AP. From (5) we see, therefore, that T is a point on the involute; and since PT is a normal to the involute, its tangent at T must be TM. But from (6) TM is constant; hence the involute is a curve such that the length of the tangent between its point of contact and a fixed right line, Ox, is constant. The involute is, therefore, a tractory (see p. 271).

The tension at any point of the catenary is equal to the weight of a portion of the string whose length is equal to the ordinate of the point.

For, by (2), if T is the tension at P,

$$T = w \,.\, s \operatorname{cosec} \theta = w \,.\, PT \operatorname{cosec} PMT$$

$$= w \,.\, PM = w \,.\, y. \tag{17}$$

It follows from this that if a uniform inextensible string hangs freely over any two smooth pegs, the vertical portions which hang over the pegs must each terminate on the horizontal axis of the catenary.

In the catenary the length of the radius of curvature at any point is equal to the length of the normal between that point and the directrix.

For, differentiating (4) and observing that $\frac{ds}{d\theta} = \rho$, where ρ is the radius of curvature at P, we have $1 + \frac{s^2}{c^2} = \frac{\rho}{c}$, $\therefore\ \rho = \frac{y^2}{c}$, and the length of the normal between P and the axis of x is $y \sec \theta$ or $\frac{y^2}{c}$.

From this follows the result that *if the catenary revolves round its directrix, Ox, it will generate a surface at each point of which the radii of curvature of the two principal sections are equal and opposite.* For, at the point P on the surface the two principal sections are those made by the plane of the figure (i.e., the plane of the revolving curve) and by a plane through the normal to the curve at P perpendicular to the plane of the figure, and the radii of curvature of these sections are the radius of curvature of the revolving curve, and the length of the normal between P and the axis of revolution.

Again, *given any two points, A, B, and any line, L, coplanar with A and B, the curve of given length passing through A and B whose centroid is at a maximum distance from the line L is a catenary.* For, imagine the curve of given length to be formed by a uniform flexible inextensible string, and let the plane of L, A, B be placed so that the line L is horizontal and above both A and B or above one of them. The extremities of the string being fixed at A and B, if the string is displaced in any manner, we know that it will return to the catenary form. But (Art. 121) it will always return to the form in which its centre of mass is in the lowest position that it can occupy; therefore the centre of mass of the catenary form is lower than that of any other form into which the string may be altered. Another property is this—*if a parabola rolls along a right line, without sliding, the locus of its focus is a catenary.* If a rough curve in the form of an inverted catenary with its axis AO (Fig. 231) vertical is fixed in a vertical plane, and a uniform rectangular board whose height is $2c$ is placed with its centre of gravity vertically over the vertex A of the curve, this board will be in equilibrium in all positions into which it can be displaced by rolling along the curve. For, when it rolls so that its side becomes the tangent TP, its centre of gravity will be M, since $MT = c$, and M is vertically over the point P of contact of the board with the curve, and if the angle TMP is less than the angle of friction, the board will rest in this position.

It will be readily seen that the differential equation of the catenary can be written in the form $c^2 \frac{d^2y}{dx^2} = y$, and that the area $OAPM =$ twice the area of the triangle PTM.

It is well to observe that if any mass is suspended from a given point of a catenary, the continuity of the curve ceases at that point, and the portions of the string at opposite sides of the point must be treated as branches of two distinct catenaries.

185.] **Any String under the action of Gravity.** Let Fig. 231 represent a string whose mass per unit length varies according to any law, with any two points in it fixed and hanging in equilibrium under the action of gravity. Let w be the weight per unit length at any point of the string. This quantity we may imagine as measured thus: at any point take an indefinitely small length, Δs, of the string, and let ΔW be the weight of this portion; then $\frac{\Delta W}{\Delta s}$ is, when Δs is taken infinitely small, the weight of the string at the point per unit length. In the notation of the Differential Calculus, $w = \frac{dW}{ds}$.

Consider, as before, the separate equilibrium of the portion AP. The weight of this portion is $\int_0^s w\,ds$, and the string must assume such a form that this force passes through Q, the point of intersection of the tangents at the ends A and P. Denoting by θ the angle made by PQ with the horizon, and by τ the tension at A—the tangent at this point being supposed horizontal—we have

$$T\cos\theta = \tau \tag{1}$$

$$T\sin\theta = \int_0^s w\,ds. \tag{2}$$

If the end A is such that the tangent at it is not horizontal, let T_0 and θ_0 be the values of T and θ at it; then in (1) we have merely to put $T_0\cos\theta_0$ instead of τ. From (1) and (2)

$$\tau\tan\theta = \int_0^s w\,ds. \tag{3}$$

The differential equation of the curve is obtained by differentiating (3) with respect to s; thus,

$$\tau\frac{d}{ds}(\tan\theta) = w. \tag{4}$$

Taking vertical and horizontal lines through any fixed point as axes of y and x, respectively, and denoting the co-ordinates of P by (x, y), the last equation becomes

$$\tau\frac{d}{ds}\left(\frac{dy}{dx}\right) = w, \tag{5}$$

which determines the law of variation of the mass per unit length at each point so that the string should hang in the form of any given curve. In (5) the differentiations may be expressed in terms of x alone or y alone. Thus

$$\tau \frac{\frac{d^2y}{dx^2}}{\frac{ds}{dx}} = w, \tag{6}$$

or

$$\tau \frac{\frac{dp}{dy}}{\sqrt{1+p^2}} = w, \tag{7}$$

where $p \equiv \dfrac{dy}{dx}$.

Cor. In order that the string should assume the form of a right line, $\tan\theta$ must be the same at all points, $\therefore\ w = 0$, as is *à priori* evident.

186.] **Catenary of Uniform Strength.** Suppose the string to be composed of homogeneous material and to have a cross-section of variable area, in such a way that at every point the area of this section is proportional to the magnitude of the tension at the point; then the curve is called a *Catenary of uniform strength.* To find the form of the string, let w now stand for the weight of the material per unit volume, and let σ be the area of the cross-section at any point P. Then the weight of a length Δs at P is $w\sigma\Delta s$, and the weight per unit length is $w\sigma$. Also let $T = p\sigma$, where p is a constant (evidently a force per unit area). Then the weight per unit length is $\dfrac{wT}{p}$, or $\dfrac{w\tau}{p\cos\theta}$; and substituting this in (4) instead of w, we have

$$\frac{d}{ds}(\tan\theta) = \frac{w}{p}\sec\theta, \tag{1}$$

$$\therefore\ \frac{d}{dx}(\tan\theta) = \frac{w}{p}\sec^2\theta, \tag{2}$$

and putting $\sec^2\theta = 1+\tan^2\theta$, we have by integration

$$\theta = \frac{wx}{p} + C, \tag{3}$$

where C is a constant. If the axes are taken through the lowest point, A, of the curve, $\theta = 0$ and $x = 0$ at A, $\therefore C = 0$,

$$\therefore \frac{dy}{dx} = \tan\frac{wx}{p} \tag{4}$$

from (3); and the integral of this again is

$$\frac{wy}{p} = -\log_e \cos\frac{wx}{p} + C',$$

where C' is a constant, which is zero $\because x = 0,\ y = 0$ at A. Hence

$$y = \frac{p}{w}\log_e \sec\frac{wx}{p}, \tag{5}$$

which is the equation of the curve referred to axes through its vertex.

The equation of the curve in the differential form (1) leads to a very remarkable expression*. Thus, denoting $\frac{p}{w}$ by a, a constant linear magnitude, (1) gives, since $\rho = \frac{ds}{d\theta}$,

$$\cos\theta = \frac{a}{\rho}, \tag{6}$$

which by differentiation again with respect to s, and substitution from (6), gives

$$\sqrt{\rho^2 - a^2} = a\frac{d\rho}{ds}. \tag{7}$$

Integrate this, and observe that (6) gives $\rho = a$ at the lowest point, A. Then

$$\log_e \frac{\rho + \sqrt{\rho^2 - a^2}}{a} = \frac{s}{a}, \tag{8}$$

or

$$\rho = \frac{a}{2}\left(e^{\frac{s}{a}} + e^{-\frac{s}{a}}\right) \tag{9}$$

$$= a\cosh\frac{s}{a}, \tag{10}$$

which is of exactly the same form as the Cartesian equation of the Common Catenary, ρ and s taking the places of y and x.

Given the whole weight, W, of the catenary, and the span, $2b$, express the area of the cross-section at each point so that there shall be a constant tension, p, per unit of area of cross-section.

Here $W = w\int\sigma ds$, where w is still the weight per unit volume of the material. But $\sigma = \frac{T}{p} = \frac{\tau}{p}\sec\theta$,

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* First noticed, I believe, in the first edition of this work.

$$\therefore\ W = \frac{\tau w}{p}\int \sec\theta\, ds = \frac{2\tau w}{p}\int_0^b \sec^2\theta\, dx$$

$$= \frac{2\tau w}{p}\int_0^b \sec^2\frac{wx}{p}\, dx, \text{ by (3)}$$

$$= 2\tau \tan\frac{wb}{p}, \tag{11}$$

which determines τ, and since $\sigma = \frac{\tau}{p}\sec\theta$,

$$\sigma = \frac{W}{2p}\cot\frac{wb}{p}\sec\theta. \tag{12}$$

But by (6) we have $\sec\theta\,\frac{p}{a} = \cosh\frac{s}{a}$, by (10); hence

$$\sigma = \frac{W}{2p}\cot\frac{wb}{p}\cosh\frac{ws}{p}, \tag{13}$$

which determines the area of the cross-section at each point.

187.] **The Parabola of Suspension Bridges.** Suppose a string to be attached to two fixed points, and let each element of its length be acted on by a force in a constant direction, the magnitude of the force being proportional to the projection of the element on a line perpendicular to the direction of the force. Then it can be shown geometrically that the figure of the string is that of a parabola.

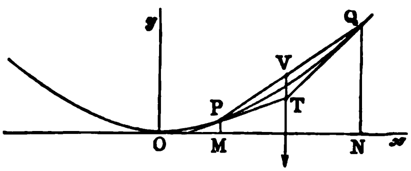

Fig. 232.

Let Oy (Fig. 232) be the direction opposite to that of the force on each element; Ox a tangent to the curve, perpendicular to this direction; P and Q any two points on the string, the tangents at them being PT and QT; PM and QN perpendiculars on Ox. Consider the separate equilibrium of the portion PQ. The forces acting on it are the tensions in the directions TP and TQ, and the resultant of the parallel forces on the elements of PQ. This resultant must pass through T, and it also passes through the middle point of MN, since its constituent forces are all proportional to the elements of the line MN. Hence, drawing TV parallel to Oy, and meeting PQ in V, the point V must bisect the right line PQ.

The curve of equilibrium of the string is therefore such that *a right line drawn from the point of intersection of any two tangents parallel to a fixed direction bisects the chord joining their points of contact.*

This well-known property identifies the curve with a parabola.

It is to be observed that the acting forces in this case are not a conservative system. Hence the function V (see sequel) does not exist.

Examples of Strings under action of Gravity.

1. A uniform string of length l hangs over two fixed points, which are in a horizontal line; from its middle point is suspended by one end another string of equal thickness and length l'. Supposing each of the two tangents of the former string at its middle point to make an angle θ with the vertical, to find the distance between the two fixed points, and to show that θ can never exceed a certain value. (Walton's *Mechanical Problems*, p. 123.)

Let the fixed points be P and Q (Fig. 233), $RQCPM$ the string hanging over them, CD the string of length l' suspended from C, the middle point of the first string, and $2d$ the distance PQ.

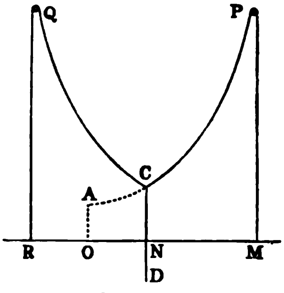

Fig. 233.

Then (Art. 184) the arcs PC and QC belong to two distinct catenaries. Suppose the semi-catenary to which PC belongs to be completed, and let A be its lowest point. Then if the portion AC were supplied to the string CPM, and the point A fixed, the string CD and the portion CQR might both be removed, and we should have the string APM hanging in equilibrium. Hence (Art. 184) PM terminates on the horizontal axis of this catenary. The same remarks apply to the portion CQR, and since the two portions CPM and CQR are exactly similar, it follows RM is the horizontal axis of the catenary AP.

We shall next prove that $AC = \frac{1}{2} CD = \frac{l'}{2}$.

Let w be the weight per unit length of each of the strings; let T be the common tension of the portions CP and CQ at C. Then resolving vertically for the equilibrium of the point C,

$$2T\cos\theta = wl'.$$

But $T = w \, . \, CN$ (Art. 184), N being the point in which CD meets the axis. Hence $2CN\cos\theta = l'$; but it is evident from Fig. 231 that $CN\cos\theta = AC$; therefore $AC = \frac{1}{2}l'$.

Again, c being the parameter of the catenary, we have $c = AC \times \tan\theta$; therefore
$$c = \tfrac{1}{2} l' \tan\theta. \tag{1}$$

Also, denoting ON by x, O being the origin of the catenary, we have
$$AC = \frac{c}{2}\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right),$$
or
$$\frac{l'}{2} = \frac{l'}{4}\tan\theta\left(e^{\frac{2x}{l'}\cot\theta} - e^{-\frac{2x}{l'}\cot\theta}\right),$$
$$\therefore \quad 2\cot\theta = e^{\frac{2x}{l'}\cot\theta} - e^{-\frac{2x}{l'}\cot\theta}.$$

Squaring both sides of this equation, adding 4 to each side, and taking the square root, we have
$$2\operatorname{cosec}\theta = e^{\frac{2x}{l'}\cot\theta} + e^{-\frac{2x}{l'}\cot\theta};$$
which, by addition to the last result, gives easily
$$x = \frac{l'}{2}\tan\theta\log\cot\frac{\theta}{2}\cdot \tag{2}$$

Again,
$$AP = \frac{c}{2}\left(e^{\frac{x+d}{c}} - e^{-\frac{x+d}{c}}\right),$$
and
$$PM = \frac{c}{2}\left(e^{\frac{x+d}{c}} + e^{-\frac{x+d}{c}}\right);$$
therefore by addition we have, since $CP + PM = \frac{1}{2}l$,
$$\frac{l+l'}{2} = ce^{\frac{x+d}{c}}.$$

Substituting in this equation the values of c and x given by (1) and (2); and taking logarithms, we have
$$2d = l'\tan\theta\log\left(\frac{l+l'}{l'}\cdot\frac{\tan\frac{1}{2}\theta}{\tan\theta}\right), \tag{3}$$
which is the required distance between P and Q.

Since d cannot be negative, the expression whose logarithm is taken in (3) must be >1. Hence $(l+l')\tan\frac{1}{2}\theta > l'\tan\theta$; and substituting for $\tan\theta$ in terms of $\tan\frac{1}{2}\theta$, we find the limiting value of θ given by the equation
$$\tan^2\frac{\theta}{2} = \frac{l-l'}{l+l'}\cdot$$

2. A uniform chain hangs over two smooth pegs in the same horizontal line, and at a given distance apart; find the length of the chain when the pressure on each peg is a minimum.

Let P and Q be the pegs, $2a$ the distance between them, l the length of the chain, θ the angle which the tangent to the chain at P makes with the vertical, PM the portion which hangs over the peg P, and C the lowest point of the chain.

Then $CP+PM = ce^{\frac{a}{c}}$ (by adding the values of CP and PM), or

$$\frac{l}{2} = ce^{\frac{a}{c}}, \qquad (1)$$

an equation which determines l in terms of c.

Again, $CP = c\cot\theta$, and $PM = c\operatorname{cosec}\theta$, therefore by addition

$$\tan\frac{\theta}{2} = e^{-\frac{a}{c}}. \qquad (2)$$

Now, the pressure on the peg P is the resultant of two equal tensions, one along PM and the other along the tangent to the chain at P. Hence, if R denote the pressure, and T the tension at P,

$$R = 2T\cos\frac{\theta}{2}.$$

Substituting for T the value $\frac{1}{2}wc\,(e^{\frac{a}{c}}+e^{-\frac{a}{c}})$, and for $\cos\frac{\theta}{2}$ its value obtained from (2), we have

$$R = wc\,(e^{\frac{2a}{c}}+1)^{\frac{1}{2}}. \qquad (3)$$

Now, c must be determined so that R is least; hence $\frac{dR}{dc} = 0$, and we obtain easily

$$e^{\frac{2a}{c}} = \frac{c}{a-c}, \qquad (4)$$

for the determination of c in terms of a; l is then known from (1).

3. Find the law of variation of the mass per unit length at each point of a string acted upon by gravity only and suspended from two fixed points in order that it may hang in the form of a semicircle whose diameter is horizontal.

Let $AB\,(=2a)$ be the horizontal diameter, O the centre of the circle, P any point on the curve, θ the angle between OP and the vertical. Then by (4), Art. 185,

$$w = \frac{\tau}{a}\sec^2\theta$$

$$= \frac{\tau a}{y^2},$$

where y is the distance of P from the diameter AB.

4. A heavy string whose mass per unit length is variable, suspended from two fixed points, hangs under the action of gravity in a curve whose intrinsic equation is $s = f(\theta)$, the lowest point being the origin; prove that the mass per unit length at any point varies inversely as $\cos^2\theta\,.\,f'(\theta)$.

5. Prove that the area of the normal section at any point of the catenary of uniform strength is proportional to the radius of curvature.

6. Find the law of variation of mass per unit length in order that a string may hang in the form of a parabola.

Ans. The mass is proportional to the horizontal projection of the curve per unit length of arc at any point. (Compare p. 352.)

7. If a string hangs in the form of an ellipse whose axis major is horizontal, prove that its weight per unit length at any point is

$$\tau \frac{b^3}{ab' y^2},$$

y being the depth of the point below the axis major, and b' the length of the semi-conjugate diameter corresponding to the point.

8. A telegraph line is constructed of No. 8 iron wire whose mass is 7·3 lbs. per 100 feet of length; the distance between the posts is 150 feet, and the wire sags 1 foot in the middle; show that it is screwed up to a tension of about 205 lbs. weight.

9. One extremity of a uniform string is attached to a fixed point, and the string rests partly on a smooth inclined plane; prove that the directrix of the catenary determined by the portion which is not in contact with the plane is the horizontal line drawn through the extremity which rests on the plane.

10. If in the last example i is the inclination of the plane, a the inclination of the tangent at the fixed extremity, and l the whole length of the string, prove that the length of the portion in contact with the plane is

$$\frac{l \cos a}{\cos i \cos (a - i)}.$$

11. A heavy uniform chain passes over a fixed smooth peg, one end hanging freely while a portion of the chain of length $2k$ rests on a smooth inclined plane of inclination i. If the length of the free part is k, show that the whole length of the chain is $4k \cos^2 i$, and that i cannot be $> \frac{\pi}{6}$.

12. Given two smooth pegs in a horizontal line, find the least length of a uniform heavy string which will rest over them.

Ans. If $2a$ is the distance between the pegs and e the Napierian base, the least length is $2ae$, or about $2{\cdot}71828a$.

13. A uniform heavy inextensible string rests partly on a rough horizontal table, and partly over a smooth peg, B, fixed at a given height, h, above the edge of the table, a portion BC of the string hanging vertically over the peg and past the edge of the table; find the length of the hanging portion, BC, so that its weight may just suffice to drag the string off the table, the string and the peg lying in the same vertical plane.

Ans. If l is the whole length of the string, μ the coefficient of friction, and x the length which hangs below the edge of the table,

$$[\mu (l-h) - (1 + \mu) x]^2 = \mu^2 (h^2 + 2hx),$$

only one value of x being admissible.

188.] **Gravitation and Absolute Measure of Force.** In the discussion of the forms assumed by strings under the action of gravity we have determined the *weight* per unit length of a string at each point in order that it should hang in a particular form. When we are not dealing exclusively with gravity as the force acting, it may be desirable to measure forces in *absolute measure*. Suppose, to fix ideas, that we adopt the C. G. S. absolute system. In this forces are expressed in dynes, and mass in grammes. If we have a mass of m grammes, the number expressing the weight of this mass in dynes would be about $981m$, or, more accurately, $g \times m$, where g stands for the acceleration, in centimètres per second per second, of a particle falling freely in vacuo near the Earth's surface at the place where we are. The number expressing the weight of the body in gravitation measure would be simply m—the same as that which expresses its mass—if the weight of 1 gramme is our gravitation unit of force. In the British absolute system forces are measured in poundals, and masses in pounds; and if we have a body of m pounds mass, its weight in gravitation measure is simply m, if the weight of 1 pound is the unit of force; while its weight expressed in poundals is about $32\,m$, or, more accurately, $g \times m$, where g is about 32 feet per second per second.

In the previous articles we may, if we choose, use w to express the mass of the string per unit length, while regarding its weight as $g \times w$ in absolute units. The letter w, however, obviously suggests weight, (i.e. force), so that mass is usually represented by the letter m. In many of the following Articles we shall denote the mass of a string per unit length at any point by m, (grammes or pounds) and assume forces as expressed in absolute measure (dynes, or poundals).

189.] **Tangential and Normal Resolutions.** Let AB (Fig. 234) represent a flexible inextensible string in equilibrium under the action of any system of coplanar forces applied continuously throughout the string. Then the force acting on it per unit mass at any point of the string will, in the general case, be expressed as a function of the co-ordinates of this point and their differential coefficients with respect to the arc. Thus, if the co-ordinates of P are (x, y), the plane of the string and forces being taken as that of xy, the external force exerted per unit mass at P will be of the form $\phi\,(x, y)$, and therefore the force exerted on dm units of mass at P will be

$$\phi\,(x,y)\,dm, \quad \text{or} \quad F\,.\,dm.$$

Suppose then that we consider the equilibrium of the element PQ of the string apart from the rest of the string:—Let the mass of the string per unit length at P be m and let ds be the length of PQ. Then the mass of the element is $m\,ds$, and the external force acting on it is

$$mF\,.\,ds\,;$$

and, in addition, it is acted upon by two tensions, T and $T+dT$, along the tangents at its extremities P and Q. These three forces must, of course, meet in a point.

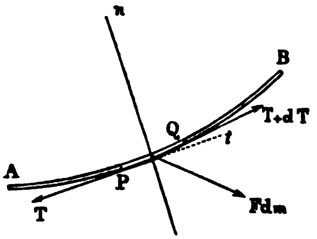

Fig. 234.

Let Pt and Pn be the tangent and normal at P; let $d\theta$ be the (very small) angle between the tangent at P and that at Q; and let ϕ be the angle between Fdm and Pt. Then, resolving forces along the tangent for the equilibrium of PQ,

$$(T+dT)\cos d\theta + mF\cos\phi\,ds - T = 0\,;$$

but $\cos d\theta = 1$, neglecting $(d\theta)^2$; therefore this equation gives

$$\frac{dT}{ds} + mF\cos\phi = 0, \tag{1}$$

which means that the rate of variation of the tension per unit length at any point is numerically equal to the tangential component of the external force per unit length.

Again, resolving along the normal,

$$(T+dT)\sin d\theta - mF\sin\phi\,ds = 0\,;$$

or since ρ, the radius of curvature at P, is equal to $\dfrac{ds}{d\theta}$,

$$\frac{T}{\rho} - mF\sin\phi = 0, \tag{2}$$

which means that the curvature of the string at any point is equal to the normal external force per unit length divided by the tension at the point.

190.] **General Equations of Equilibrium.** Let the external force, F, per unit mass at P be resolved into two components, X and Y, parallel to any pair of fixed rectangular axes. Then the components of force acting on the element PQ parallel to the axes are $mXds$ and $mYds$. Also the components of the tension acting at the end P are

$$-T\frac{dx}{ds}, \quad -T\frac{dy}{ds},$$

each measured in the positive sense of the corresponding axis. The components of tension at P are affected with negative signs, since, when the element PQ is considered apart from the rest of the string, the tensions at P and Q will manifestly give components along the axis of x in *opposite* senses; and similarly along the axis of y.

These components of tension at any point will be functions of the position of the point on the string, i.e. functions of the length of the string measured up to the point considered from any origin-point, A, on the string. If the length of the string $AP = s$, we shall therefore have

$$T\frac{dx}{ds} = f(s),$$

and the component of the tension at Q is therefore $f(s + ds)$, or

$$f(s) + f'(s).ds + f''(s)\frac{ds^2}{1.2} + \ldots,$$

or, again, $$T\frac{dx}{ds} + \frac{d}{ds}\left(T\frac{dx}{ds}\right).ds + \frac{d^2}{ds^2}\left(T\frac{dx}{ds}\right).\frac{ds^2}{1.2} + \ldots.$$

Hence for the equilibrium of the element PQ, resolving forces parallel to the axis of x,

$$T\frac{dx}{ds} + \frac{d}{ds}\left(T\frac{dx}{ds}\right).ds + \ldots + mXds - T\frac{dx}{ds} = 0;$$

or, rejecting the two terms which cancel, dividing out by ds, and then diminishing ds indefinitely,

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) + mX = 0. \qquad (1)$$

Similarly $$\frac{d}{ds}\left(T\frac{dy}{ds}\right) + mY = 0. \qquad (2)$$

These are the general equations of equilibrium of the string.

The value of T may be deduced in various ways from these equations. Thus, performing the differentiations,

$$T\frac{d^2x}{ds^2} + \frac{dT}{ds}\cdot\frac{dx}{ds} + mX = 0, \tag{3}$$

$$T\frac{d^2y}{ds^2} + \frac{dT}{ds}\cdot\frac{dy}{ds} + mY = 0. \tag{4}$$

Multiply (3) and (4) by $\frac{dx}{ds}$ and $\frac{dy}{ds}$, respectively, and add; then, remembering that, since $\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1$, we have by differentiation $$\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} = 0,$$

it follows that $$\frac{dT}{ds} + m\left(X\frac{dx}{ds} + Y\frac{dy}{ds}\right) = 0; \tag{5}$$

$$\therefore \quad T = C - \int m(Xdx + Ydy), \tag{6}$$

which is precisely the same as (1) of last Article.

Another expression for T can be deduced from (1) and (2). They give by integration

$$T\frac{dx}{ds} = A - \int mXds, \tag{7}$$

$$T\frac{dy}{ds} = B - \int mYds. \tag{8}$$

Squaring and adding, we have

$$T^2 = (A - \int mXds)^2 + (B - \int mYds)^2, \tag{9}$$

A and B being the constants of integration, which must be determined by a knowledge of the tension at some particular points.

Again, by multiplying (3) and (4) by $\frac{d^2x}{ds^2}$ and $\frac{d^2y}{ds^2}$, respectively, and adding, we obtain

$$\frac{T}{\rho^2} + m\left(X\frac{d^2x}{ds^2} + Y\frac{d^2y}{ds^2}\right) = 0,$$

which is the same as (2) of Art. 189, since the ection-cosines of the radius of curvature are $-\rho\frac{d^2x}{ds^2}$, $-\rho\frac{d^2y}{ds^2}$.

The curve formed by the string is found by eliminating T from (7) and (8); hence its equation is

$$(A - \int mXds)\frac{dy}{ds} - (B - \int mYds)\frac{dx}{ds} = 0.$$

If the external force at every point of the string is normal to its direction, the tension is constant throughout, as at once appears from (5); for $X\frac{dx}{ds}+Y\frac{dy}{ds}$ is the tangential component of the external force per unit mass. This is the case when, for example, the string is stretched over any smooth curve, and acted on by no force except the reaction of the curve and two terminal tensions (which must be equal). Thus we have proved the truth of our assumption in Art. 33.

191.] **Catenary of Uniform Strength in General.** The force per unit mass at any point of a string having any components X, Y, if the area of the normal section, σ, at this point is proportional to T, the tension at the point, the catenary will be of uniform strength. If then k is the mass of the substance of the string per unit volume at any point, $m = k\sigma$ in the previous equations, and if we put $\sigma = \mu T$, where μ is a constant $\left(\text{of the nature }\frac{\text{area}}{\text{force}}\right)$, into (3) and (4) of Art. 190, we have

$$T\left(\frac{d^2x}{ds^2}+k\mu X\right)+\frac{dT}{ds}\frac{dx}{ds}=0,$$

$$T\left(\frac{d^2y}{ds^2}+k\mu Y\right)+\frac{dT}{ds}\frac{dy}{ds}=0,$$

for the catenary of uniform strength, whose equation is found by eliminating T from these equations. This is done by simple division, and we have

$$\frac{d^2x}{ds^2}\frac{dy}{ds}-\frac{d^2y}{ds^2}\frac{dx}{ds}=k\mu\left(Y\frac{dx}{ds}-X\frac{dy}{ds}\right).$$

Remembering that, if ϕ is the angle made by the tangent with the axis of x, we have

$$\frac{dx}{ds}=\cos\phi,\quad \frac{dy}{ds}=\sin\phi,\quad \rho=\frac{ds}{d\phi},$$

this equation is
$$-\frac{1}{\rho}=k\mu N,$$

where N is the normal force per unit mass measured along the normal in the sense of ρ; and this equation could have been at once obtained from (2) of Art. 189.

Consider the particular case in which the applied force has a constant direction at all points in the string; and take this direction as that of the axis of y. Hence

$$T = \tau\frac{ds}{dx}, \text{ and } \frac{d}{ds}\left(\frac{dy}{dx}\right) + \mu k Y'\frac{ds}{dx} = 0,$$

where Y' is the force per unit mass at the point (x, y). Comparing these with the equations for a string of constant section acted on by parallel forces whose intensity is Y at the point (x, y), we see that the two curves will be the same provided that

$$Y' = \frac{1}{\mu} Y \frac{dx}{ds}.$$

Thus, if $Y' = g\dfrac{dx}{ds}$, for which law (see Art. 187) a uniform catenary would hang in a parabola, the catenary of uniform strength is the common catenary.

192.] **String acted on by a Central Force.** When the lines of action of the forces applied to the various elements of the string pass all through the same point, the force acting on the string is said to be *central*, and this point is called the *centre of force.* It is easy to prove that in this case the string must lie in a plane passing through the centre of force. For (Art. 189) the plane of the tangents at P and Q must contain the centre of force; and since two consecutive osculating planes have a tangent line to the string common, these two planes, having in addition a point (the centre of force) common, must be identical. Hence the osculating plane is the same at all points; or the string must lie wholly in one plane.

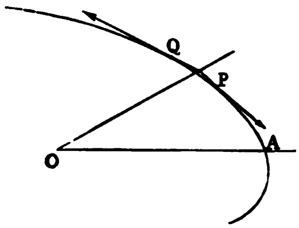

Fig. 235.

To find the form assumed by a string acted on by a given central force.

Let O (Fig. 235) be the centre of force (supposed repulsive), PQ an element of the string whose equilibrium is considered apart, r the radius vector OP, θ the angle POA between OP and a fixed initial line, s the length of the arc AP, and p the perpendicular from O on the tangent at P. Then, for the equilibrium of the element PQ, taking moments about O, we have

moment of tension at P = moment of tension at Q;

or

$$Tp = Tp + d(Tp),$$

$$\therefore\ Tp = h, \tag{1}$$

where h is a constant*.

Denote the tensions at P and Q by T and $T+dT$ respectively.

Resolve the forces acting on PQ along the tangent at P, let m = mass per unit length at P, and let the central force be $mFds$. Then this force passes through the point of intersection of tangents at P and Q, and the cosine of the angle between its direction and the tangent at P is $-\dfrac{dr}{ds}+\epsilon$, where ϵ is indefinitely small. In the equation of resolution the component of $mFds$ is

$$mFds\left(-\frac{dr}{ds}+\epsilon\right),$$

so that ϵ may be neglected, and we have

$$dT = -mFdr. \tag{2}$$

Equations (1) and (2) determine the form of the curve.

If the central force is attractive, the sign of F must be changed in (2), and the curve of equilibrium will be convex towards O.

It is usual in the discussion of central forces to denote $\dfrac{1}{r}$ by u. Making this substitution, and eliminating T from the above equations, we have

$$\frac{mF}{u^2}du = hd\left(\frac{1}{p}\right). \tag{3}$$

But (Williamson's *Differential Calculus*, Chap. XII),

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2.$$

Hence, denoting $\dfrac{mF}{u^2}$ by $\phi(u)$, and $\int\phi(u)du$ by $\phi_1(u)$, an arbitrary constant being implied in $\phi_1(u)$, we have from (3)

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{1}{h^2}\{\phi_1(u)\}^2. \tag{4}$$

It is often more convenient to retain a differential equation of the second order for u†. Differentiating (4) we have, dividing out by $\frac{du}{d\theta}$, and remembering that $\phi_1'(u) = \phi(u)$,

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* Of course this proof holds whether the portion PQ is an element of length or a portion of any length, however great.

† This method of treating the equilibrium of a string acted on by a central force is taken from a paper by Professor Townsend in the *Quarterly Journal of Pure and Applied Mathematics*, 1874.

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{h^2}\phi_1(u).\phi(u). \tag{5}$$

Now, since the integration of (4) gives u in terms of θ, and introduces an arbitrary constant in addition to that already involved in $\phi_1(u)$, we see that the solution of the problem involves only *two* arbitrary constants. But (5) will require two integrations to express u, and each integration will introduce an arbitrary constant. Hence it appears that in this way we get *three* arbitrary constants, instead of two. These three are, however, easily connected, since the values of u and $\left(\frac{du}{d\theta}\right)^2$ given by the complete integral of (5) must satisfy (4) for all values of u.

As an example, let it be required to discuss the form of a string of uniform section and density when the central repulsive force varies inversely as the square of the distance. In this case m is constant, and $F = \mu' u^2$, μ' being a constant which obviously denotes the magnitude of the force per unit mass of matter at the unit distance from the centre of force.

Hence we have, putting $m\mu' = \mu$,

$$T = C + \mu u,$$

C being a constant. If T_0 denote the tension at a point A of the string whose distance from the centre is $\frac{1}{a}$, we have, evidently,

$$\begin{aligned} T &= T_0 - \mu a + \mu u \\ &= \mu(u+c), \text{ suppose.} \end{aligned}$$

Hence,

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{\mu^2}{h^2}(u+c)^2, \tag{6}$$

which gives, by differentiation,

$$\frac{d^2u}{d\theta^2} + \left(1 - \frac{\mu^2}{h^2}\right)u - \frac{\mu^2}{h^2}c = 0. \tag{7}$$

First, suppose that $\frac{\mu^2}{h^2} < 1$, and denote $1 - \frac{\mu^2}{h^2}$ by λ^2. Then this equation becomes $\frac{d^2u}{d\theta^2} + \lambda^2\left(u - \frac{1-\lambda^2}{\lambda^2}c\right) = 0$,

the integral of which is

$$u = \frac{1-\lambda^2}{\lambda^2}c + A\cos\lambda(\theta - a),$$

A and α being the constants of integration. Substituting this value of u in (6), we have $A = \frac{\sqrt{1-\lambda^2}}{\lambda^2}c$, and therefore

$$u = \frac{1-\lambda^2}{\lambda^2}c\left\{1 + \frac{1}{\sqrt{1-\lambda^2}}\cos\lambda(\theta-\alpha)\right\}. \qquad (8)$$

The value of α is found by putting $u = a$ and $\theta =$ the angle belonging to the point A.

When $\theta = \alpha$, $\frac{du}{d\theta} = 0$, and there is an apse. If the initial line be taken through the apse, and T_0 and a belong to this point, we have $c = \frac{T_0}{\mu} - a = (\frac{h}{\mu} - 1)a$, and (8) assumes the simple form

$$u = \frac{a}{1+\frac{\mu}{h}}\left(\frac{\mu}{h} + \cos\lambda\theta\right), \qquad (9)$$

which differs from the focal polar equation of a conic in having the angle multiplied by a number, λ, less than unity.

If $\frac{\mu^2}{h^2} > 1$, we must put $\frac{\mu^2}{h^2} - 1 = \lambda^2$, and putting $\mu a - T_0 = \mu c$, equation (6) becomes $\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{\mu^2}{h^2}(u-c)^2$,

which gives
$$u = \frac{1+\lambda^2}{\lambda^2}c + Ae^{\lambda\theta} + Be^{-\lambda\theta}, \qquad (10)$$

the constants A and B being connected by the equation

$$AB = \frac{1+\lambda^2}{4\lambda^4}c^2 \text{ by } (4).$$

Equation (10) can obviously be written

$$u = \frac{1+\lambda^2}{\lambda^2}c + \sqrt{AB}\left(\sqrt{\frac{A}{B}}e^{\lambda\theta} + \sqrt{\frac{B}{A}}e^{-\lambda\theta}\right),$$

or
$$u = \frac{1+\lambda^2}{\lambda^2}c\left\{1 + \frac{1}{2\sqrt{1+\lambda^2}}\left[e^{\lambda(\theta-\alpha)} + e^{-\lambda(\theta-\alpha)}\right]\right\}. \qquad (11)$$

When $\theta = \alpha$, there is an apse, and if the initial line be taken through the apse, we have, in the same manner as before,

$$u = \frac{a}{\frac{\mu}{h}+1}\left\{\frac{\mu}{h} + \frac{e^{\lambda\theta}+e^{-\lambda\theta}}{2}\right\}. \qquad (12)$$

If $\frac{\mu}{h} = 1$, both (9) and (12) give $u = a$, a constant; and the figure of equilibrium is a circle.

out by $\frac{du}{d\theta}$, and remembering that $\phi_1'(u) = \phi(u)$,

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{h^2}\phi_1(u).\phi(u). \tag{5}$$

Now, since the integration of (4) gives u in terms of θ, and introduces an arbitrary constant in addition to that already involved in $\phi_1(u)$, we see that the solution of the problem involves only *two* arbitrary constants. But (5) will require two integrations to express u, and each integration will introduce an arbitrary constant. Hence it appears that in this way we get *three* arbitrary constants, instead of two. These three are, however, easily connected, since the values of u and $\left(\frac{du}{d\theta}\right)^2$ given by the complete integral of (5) must satisfy (4) for all values of u.

As an example, let it be required to discuss the form of a string of uniform section and density when the central repulsive force varies inversely as the square of the distance. In this case m is constant, and $F = \mu' u^2$, μ' being a constant which obviously denotes the magnitude of the force per unit mass of matter at the unit distance from the centre of force.

Hence we have, putting $m\mu' = \mu$,

$$T = C + \mu u,$$

C being a constant. If T_0 denote the tension at a point A of the string whose distance from the centre is $\frac{1}{a}$, we have, evidently,

$$\begin{aligned} T &= T_0 - \mu a + \mu u \\ &= \mu(u + c), \text{ suppose.} \end{aligned}$$

Hence,

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{\mu^2}{h^2}(u+c)^2, \tag{6}$$

which gives, by differentiation,

$$\frac{d^2u}{d\theta^2} + \left(1 - \frac{\mu^2}{h^2}\right)u - \frac{\mu^2}{h^2}c = 0. \tag{7}$$

First, suppose that $\frac{\mu^2}{h^2} < 1$, and denote $1 - \frac{\mu^2}{h^2}$ by λ^2. Then this equation becomes

$$\frac{d^2u}{d\theta^2} + \lambda^2\left(u - \frac{1-\lambda^2}{\lambda^2}c\right) = 0,$$

the integral of which is

$$u = \frac{1-\lambda^2}{\lambda^2}c + A\cos\lambda(\theta - a),$$

A and α being the constants of integration. Substituting this value of u in (6), we have $A = \frac{\sqrt{1-\lambda^2}}{\lambda^2}c$, and therefore

$$u = \frac{1-\lambda^2}{\lambda^2}c\left\{1 + \frac{1}{\sqrt{1-\lambda^2}}\cos\lambda(\theta-\alpha)\right\}. \qquad (8)$$

The value of α is found by putting $u = a$ and $\theta =$ the angle belonging to the point A.

When $\theta = \alpha$, $\frac{du}{d\theta} = 0$, and there is an apse. If the initial line be taken through the apse, and T_0 and a belong to this point, we have $c = \frac{T_0}{\mu} - a = (\frac{h}{\mu} - 1)a$, and (8) assumes the simple form

$$u = \frac{a}{1+\frac{\mu}{h}}\left(\frac{\mu}{h} + \cos\lambda\theta\right), \qquad (9)$$

which differs from the focal polar equation of a conic in having the angle multiplied by a number, λ, less than unity.

If $\frac{\mu^2}{h^2} > 1$, we must put $\frac{\mu^2}{h^2} - 1 = \lambda^2$, and putting $\mu a - T_0 = \mu c$, equation (6) becomes $\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{\mu^2}{h^2}(u-c)^2$,

which gives $$u = \frac{1+\lambda^2}{\lambda^2}c + Ae^{\lambda\theta} + Be^{-\lambda\theta}, \qquad (10)$$

the constants A and B being connected by the equation

$$AB = \frac{1+\lambda^2}{4\lambda^4}c^2 \text{ by (4).}$$

Equation (10) can obviously be written

$$u = \frac{1+\lambda^2}{\lambda^2}c + \sqrt{AB}\left(\sqrt{\frac{A}{B}}e^{\lambda\theta} + \sqrt{\frac{B}{A}}e^{-\lambda\theta}\right),$$

or $$u = \frac{1+\lambda^2}{\lambda^2}c\left\{1 + \frac{1}{2\sqrt{1+\lambda^2}}\left[e^{\lambda(\theta-\alpha)} + e^{-\lambda(\theta-\alpha)}\right]\right\}. \qquad (11)$$

When $\theta = \alpha$, there is an apse, and if the initial line be taken through the apse, we have, in the same manner as before,

$$u = \frac{a}{\frac{\mu}{h}+1}\left\{\frac{\mu}{h} + \frac{e^{\lambda\theta} + e^{-\lambda\theta}}{2}\right\}. \qquad (12)$$

If $\frac{\mu}{h} = 1$, both (9) and (12) give $u = a$, a constant; and the figure of equilibrium is a circle.

Equation (8) can be written in the form

$$\frac{l}{r} = 1 + e\cos\lambda(\theta - a),$$

where $\lambda^2 = \frac{e^2-1}{e^2}$, and (11) can be written

$$\frac{l}{r} = 1 + e\cosh\lambda(\theta - a),$$

where $\lambda^2 = \frac{1-e^2}{e^2}$.

For the remarkable analogy between the curve of equilibrium of a flexible string and the orbit of a particle under a given force, see Professor Townsend's Paper, and Thomson and Tait's *Nat. Phil.*

193.] **Apsidal Angle.** To find the angle between the apsides in a string which, under the action of a central force, assumes a form nearly circular.

DEF. An apse is a point on a curve at which the radius vector is at right angles to the tangent.

Since the form of the string is nearly circular, u will differ from a constant value, a, by a small variable quantity, x.

Let, then, $u = a + x$. In this case $\phi_1(u) = \phi_1(a) + x\phi(a)$, neglecting higher powers of x; and $\phi(u) = \phi(a) + x\phi'(a)$. For shortness, denote $\phi_1(a)$, $\phi(a)$, and $\phi'(a)$ by ϕ_1, ϕ, and ϕ' respectively. Then (5) of last Art. becomes

$$\frac{d^2x}{d\theta^2} + a + x = \frac{1}{h^2}\{\phi\phi_1 + (\phi_1\phi' + \phi^2)x\}. \tag{1}$$

But if the string were exactly circular, x and $\frac{d^2x}{d\theta^2}$ would always $= 0$; therefore $a = \frac{\phi\phi_1}{h^2}$, or

$$\frac{1}{h^2} = \frac{a}{\phi\phi_1}. \tag{2}$$

Hence (1) becomes

$$\frac{d^2x}{d\theta^2} + \left\{1 - a\left(\frac{\phi'}{\phi} + \frac{\phi}{\phi_1}\right)\right\}x = 0. \tag{3}$$

The constant a may be chosen as the reciprocal of the radius of any circle which nearly coincides with the figure of the string; but simplicity is gained by taking it equal to the reciprocal of the radius of that circle in which the tension at each point is equal to the *mean* tension in the string.

Now in a circle of radius $\frac{1}{a}$ the tension (see (2), Art. 189) is $a\phi$;

and (2) of last Art. gives T in the curve equal to $\phi_1(u)$, and therefore the mean tension $= \phi_1$. Hence

$$a\phi = \phi_1,$$

and (3) finally becomes
$$\frac{d^2x}{d\theta^2} - \frac{a\phi'}{\phi}x = 0. \tag{4}$$

If $\frac{a\phi'}{\phi}$ be positive, the value of x in terms of θ will be exponential, and the nearly circular form becomes impossible, since the value of u increases indefinitely with θ.

For the possibility of a nearly circular form $\frac{a\phi'}{\phi}$ must be negative, and we have

$$x = A\cos\left(\sqrt{\frac{-a\phi'}{\phi}}\,\theta - a\right).$$

Hence, since at an apse $\frac{du}{d\theta} = 0$, or $\frac{dx}{d\theta} = 0$, we shall arrive at an apse whenever

$$\sin\left(\sqrt{\frac{-a\phi'}{\phi}}\,\theta - a\right) = 0,$$

and the difference between two successive values of θ which satisfy this equation is

$$\frac{\pi}{\sqrt{\frac{-a\phi'}{\phi}}},$$

which is, therefore, the angle between the apsides*.

194.] **Diagram of Tensions.** When a string is in equilibrium under the action of any specified distribution of force on its various elements of length, we may, of course, construct a force-polygon, such as that in Fig. 30, p. 42, whose sides are proportional and parallel to the resultant forces acting on the successive infinitesimal elements of arc of the string. Of course the sides of this force-polygon will not be of finite length, as they are in Fig. 30; they will, in fact, form a continuous curve—let us say, the *force-curve*. And again, there will be a pole, O (Fig. 30), from which if radii vectores are drawn to the force-curve, these radii vectores will be parallel and proportional to the tensions of the string at the corresponding points. Hence if T denotes the tension of the string at any point P, and ψ is the angle which the tangent at P makes with any fixed line through O, the polar co-ordinates of the point, Π, on the force-curve corresponding to P can be taken as

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* This investigation is taken from the paper by Professor Townsend previously referred to.

$$T, \psi,$$

and the rectangular co-ordinates of Π are $(T\cos\psi, T\sin\psi)$; so that if we denote these co-ordinates by (ξ, η), equations (7) and (8) of Art. 190 can be written

$$\xi = A - \int mXds, \tag{1}$$

$$\eta = B - \int mYds, \tag{2}$$

and if we can express each variable at the right-hand sides of these equations as a function of s alone, the equation of the force-curve, and therefore the diagram of tensions, will be obtained by eliminating s from these equations.

For any string, uniform or not, under the action of gravity or of any system of parallel forces, the force-curve is a right line parallel to the common direction, and the perpendicular from the pole O on this line represents τ, the constant component which the tension at every point has perpendicular to the direction of the parallel system.

In the case of a string acted upon by a central force, as in Fig. 235, the equations of equilibrium obtained by resolving along the tangent and normal for an element PQ are

$$dT + mFdr = 0, \tag{3}$$

$$Td\psi - mFrd\theta = 0, \tag{4}$$

where $OP = r$, $\angle POA = \theta$, $\psi =$ the angle made with OA by the tangent at P, measured in the same sense as θ.

We have obviously $\cot(\psi - \theta) = \dfrac{dr}{rd\theta}$;

$$\therefore \quad \frac{dT}{Td\psi} = -\cot(\psi - \theta), \tag{5}$$

so that the relation between T and ψ, i. e., the equation of the force-curve, is obtained by expressing θ in terms of ψ (from the equation of the curve of the string) and substituting in the right-hand side of (5), which then becomes a function of ψ alone. As an example, take the case of a uniform string acted upon by a central force $\dfrac{\mu}{r^3}$. From (3) we have

$$T = \frac{m\mu}{2r^2} + C,$$

where C is some constant. If for simplicity we consider $C = 0$ —or, in other words, if we imagine that at one end, A, of the string, distant c from the centre of force, the tension is by any external means made equal to $\frac{m\mu}{2c^2}$—we have $T = \frac{m\mu}{2r^2}$, and the equation $Tp = h$ (p. 363) gives

$$\frac{p}{r^2} = \frac{2h}{m\mu},$$

which shows the curve of the string to be a circle passing through the centre of force. If the string followed the circle all round, the tension would be infinite at the centre of force, but we may suppose the string to have two fixed ends, A and B, at any points of its length.

Hence $\psi = 2\theta - \frac{\pi}{2}$, and (5) gives

$$\frac{dT}{Td\psi} = \tan\left(\frac{\pi}{4} + \frac{\psi}{2}\right)$$

$$\therefore\ T = \frac{k}{1 - \sin\psi},$$

where k is a constant. This is the equation of the force-curve, which is, therefore, a parabola whose focus is the centre of force.

195.] **String on Smooth Plane Curve.** Consider the case of an inextensible string resting on a smooth plane curve under the action of any forces in the plane of the curve, and let Fig. 234 represent this case. Then into the equations of Art. 189 we have merely to introduce the normal reaction, Rds*, acting on the element PQ in the direction nP.

Resolving tangentially, we obtain

$$\frac{dT}{ds} + mF\cos\phi = 0. \qquad (1)$$

Resolving normally,

$$\frac{T}{\rho} - mF\sin\phi - R = 0. \qquad (2)$$

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* The student will observe that in considering the equilibrium of an element of length ds we represent the reaction of a curve on it by Rds, and the applied force by $mFds$, while we represent the tension by T, and not by Tds. The reason of this is that the tension depends merely on the cross section of the element and not on its length, while the magnitude of the reaction depends evidently on the length of the element in contact with the curve.

These are the most useful resolutions in the case of a string resting on a curve. Equations of resolution along arbitrary axes may, of course, be obtained by introducing the components of R into the general equations of Art. 190.

From (1) we obtain $T = C - \int m F \cos\phi\, ds$, C being a constant; and if we denote the general integral $\int m F \cos\phi\, ds$ by V, its value when the co-ordinates of A are substituted in V being V_0, while T_0 is the tension at A, we have

$$T = T_0 - (V - V_0).$$

We shall refer to V as the *potential* of the external forces at P.

196.] **String on Rough Plane Curve.** If the curve in the preceding Article is rough, and the string in limiting equilibrium, slipping being about to take place in the direction QP, we have merely to include among the forces acting on the element PQ a tangential force $\mu R ds$, the coefficient of friction being μ and the normal reaction $R\,ds$, as before.

Equations (1) and (2) of last Article now become

$$\frac{dT}{ds} + mF\cos\phi + \mu R = 0, \tag{1}$$

$$\frac{T}{\rho} - mF\sin\phi - R = 0. \tag{2}$$

It will be observed that the components of reaction, $R ds$ and $\mu R ds$, are in the figure represented as acting at P. In strictness, of course, they do not act at P; but in the limit the same equations will be obtained, no matter at what point between P and Q we represent them as acting.

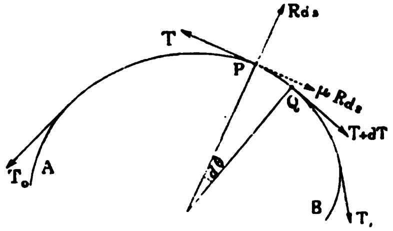

Fig. 236.

Consider the simple case in which there is no external force continuously applied throughout the string, or $F = 0$. Then these equations become

$$dT + \mu R ds = 0, \tag{3}$$

$$T d\theta - R ds = 0. \tag{4}$$

Hence $$\frac{dT}{T} + \mu d\theta = 0, \quad \therefore\ T = Ce^{-\mu\theta},$$

C being the constant of integration, and θ the angle between the tangent at the point P and the tangent at some origin point, A, on the string. If T_0 is the tension at A, we have $T = T_0$ when $\theta = 0$; therefore

$$T = T_0 e^{-\mu\theta}. \tag{5}$$

Hence, as the angle through which the string turns increases in arithmetical progression, the tension decreases in geometrical progression.

Suppose that (the weight of the string being neglected) two masses, P and Q, are suspended from the extremities of a string which passes over a fixed rough cylinder whose axis is horizontal, the string lying in a plane perpendicular to this axis; it is required to find the relation between P and Q when the equilibrium is limiting.

Let A (Fig. 236) be the point at which the portion of the string next P leaves the cylinder, and B the point at which the portion next Q leaves it.

Then from (5), by putting T_0 = weight of P and $\theta = \pi$, we have

$$Q = Pe^{-\mu\pi}, \tag{6}$$

when P is about to overcome Q. If P is on the point of ascending, the sign of μ in this equation is to be changed.

If the string makes a complete revolution and a half round the cylinder, the value of θ corresponding to Q is 3π, and we have in this case $Q = Pe^{-3\mu\pi}$. The factor $e^{-\mu\theta}$ diminishes very rapidly as the angle increases, and thus we see how it is that a small force applied at one extremity of a rope coiled several times round a fixed rough cylinder can overcome a large force applied at the other extremity—a practical example of which occurs when the small motion of a ship in harbour is stopped by a small force applied at the extremity of a rope coiled round a fixed post. For example, if $\mu = \frac{1}{2}$, $e^{\mu\pi} = 4.8$, and $Q = \dfrac{P}{4.8}$.

EXAMPLES.

1. A uniform inextensible string, acted on by gravity and by two terminal tensions, rests in contact with a smooth curve in a vertical plane; find the form of this curve so that the pressure which it exerts on the string may at every point be inversely proportional to the radius of curvature.

Let vertical and horizontal lines in the plane of the curve be taken as axes of y and x, respectively, and let the concavity of the curve be upwards.

Then, R being the pressure per unit of length at any point, and T the tension at this point, we have, by resolving along the tangent,

$$dT = mgdy,$$

mg being the weight of a unit of length of the string. Hence

$$T = T_0 + mg(y - y_0), \qquad (1)$$

T_0 and y_0 belonging to one end of the string.

Again, resolving normally,

$$Td\theta - mgdx = Rds,$$

($d\theta$ being the angle between two consecutive tangents), or

$$\frac{T}{\rho} - mg\frac{dx}{ds} = R. \qquad (2)$$

Let $R = \frac{k}{\rho}$, k being a constant. Then from (1) and (2)

$$\frac{T_0 - k + mg(y - y_0)}{\rho} = mg\frac{dx}{ds},$$

or

$$\frac{y - \lambda}{\rho} = \frac{dx}{ds}, \qquad (3)$$

denoting the numerator of the left-hand side of the previous equation by $mg(y - \lambda)$, for simplicity. To integrate (3), put

$$\frac{dx}{ds} = \frac{1}{\sqrt{1 + p^2}}, \text{ and } \rho = \frac{(1 + p^2)^{\frac{3}{2}}}{p\frac{dp}{dy}} \text{ where } p \equiv \frac{dy}{dx}.$$

The equation then becomes $\frac{pdp}{1 + p^2} = \frac{dy}{y - \lambda}$,

$$\therefore \quad 1 + p^2 = \mu^2(y - \lambda)^2,$$

μ being the constant introduced by integration.

From this equation we have

$$\frac{dy}{\sqrt{(y - \lambda)^2 - \frac{1}{\mu^2}}} = \mu dx,$$

which gives by integration

$$y - \lambda + \sqrt{(y - \lambda)^2 - \frac{1}{\mu^2}} = be^{\mu x},$$

where b is an arbitrary constant. This equation can easily be put into the form

$$y - \lambda = \frac{b}{2}e^{\mu x} + \frac{1}{2b\mu^2}e^{-\mu x}.$$

Now, any expression of the form $Ae^{\mu x} + Be^{-\mu x}$ can be put into the form

$$C\left\{e^{\mu(x+a)} + e^{-\mu(x+a)}\right\};$$

for, identifying the two expressions, we have

$$C = \sqrt{AB}, \text{ and } e^{\mu a} = \sqrt{\frac{A}{B}}.$$

Hence we have $$y-\lambda = \frac{1}{2\mu}\left\{b\mu e^{\mu x} + \frac{1}{b\mu}e^{-\mu x}\right\}$$
$$= \frac{1}{2\mu}\{e^{\mu(x+a)} + e^{-\mu(x+a)}\},$$
where $e^{\mu a} = b\mu$.

This is, of course, the equation of a common catenary whose parameter is $\frac{1}{\mu}$, and whose origin is the point $(\lambda, -a)$.

2. A uniform inextensible string, acted on by two terminal tensions, and any system of conservative forces in one plane, rests in contact with a smooth curve in this plane; if at every point the pressure against the curve is inversely proportional to the radius of curvature, then, without any change in the forces, the tension at one extremity can be so varied that the constraining curve may be removed, and the string will rest in free equilibrium.

For, if V denote the potential of the applied forces at any point, we have (Art. 195) $$T = T_0 - (V - V_0). \tag{1}$$

Again, if N denote the normal component of applied forces at any point measured towards the convex side of the curve, and R the pressure per unit of length at this point,

$$\frac{T}{\rho} = R + N. \tag{2}$$

Suppose that $R = \frac{k}{\rho}$. Then, from (1) and (2) we have

$$\frac{T_0 - k - (V - V_0)}{\rho} - N = 0. \tag{3}$$

Let us now change the terminal tension T_0 into $T_0 - k$, and investigate the pressure of the curve at the point considered above. Denoting the new pressure by R', and the new tension by T', there being no change in any of the applied forces, we have

$$T' = T_0 - k - (V - V_0),$$
$$\frac{T'}{\rho} = R' + N,$$

from which $$R' = \frac{T_0 - k - (V - V_0)}{\rho} - N;$$

but the right-hand side of this equation is zero by (3). Hence there is no pressure at any point, and the curve is one of free equilibrium. It is obvious that the last example is a particular case of this.

3. A string is kept in equilibrium in the form of a closed curve by

the action of a repulsive force tending from a fixed point, and the area of the cross-section at each point is proportional to the tension; prove that the repulsive force at any point is inversely proportional to the chord of curvature through the centre of force. (Wolstenholme's *Book of Math. Prob.*) The equations are (Art. 192),

$$Tp = h, \tag{1}$$

$$dT = -mFdr. \tag{2}$$

Now, $m \equiv k\sigma$, and by hypothesis $\sigma \propto T$, and k is constant; therefore we have $m = \mu T$, μ being a constant. Hence from (2)

$$\frac{dT}{T} = -\mu F dr. \tag{3}$$

But from (1), $dT = -\frac{h}{p^2}dp$, therefore $\frac{dT}{T} = -\frac{dp}{p}$, and we have from (3)

$$\mu F = \frac{1}{p} \cdot \frac{dp}{dr} = \frac{2}{\gamma},$$

where γ is the chord of curvature passing through the pole (see Williamson's *Diff. Cal.*, p. 296, sixth ed.).

As a particular case, we may notice that the vertical chord of curvature at any point of the catenary of uniform strength (under gravity) is constant, as the student can easily prove otherwise.

4. A heavy inextensible string rests, in limiting equilibrium, on a rough curve in a vertical plane; find the tension at any point.

Let Fig. 236 represent the string lying on the curve; let a horizontal line above the curve AB be the axis of x, and let the axis of y be drawn vertically downwards.

Then, if θ be the angle made by the tangent at any point, P, with the axis of x, mg the weight of a unit length of the string at P, and x, y the co-ordinates of P, we get by a tangential resolution (slipping being on the point of taking place from P to Q),

$$dT - \mu R ds + mg dy = 0;$$

and by a normal resolution

$$T d\theta - R ds + mg dx = 0.$$

Eliminating R, we obtain

$$\frac{dT}{d\theta} - \mu T = mg\left(\mu \frac{dx}{d\theta} - \frac{dy}{d\theta}\right)$$

$$= mg\,(\mu \cos\theta - \sin\theta)\,\rho, \tag{1}$$

where ρ is the radius of curvature at P.

This is a linear differential equation of the first order, the solution of which is (Boole's *Differential Equations*, p. 39),

$$T = e^{\mu\theta}\left\{C + \int mg\rho(\mu\cos\theta - \sin\theta)\,e^{-\mu\theta}\,d\theta\right\}, \tag{2}$$

C being a constant.

When the curve of constraint is given, ρ is known in terms of θ, and the integration may then be performed.

For example, let the string rest on a circle of radius a, one extremity being at the highest point, and free from tension.

It will be easily found that

$$\int(\mu\cos\theta-\sin\theta)\,e^{-\mu\theta}\,d\theta = \frac{e^{-\mu\theta}}{1+\mu^2}\{2\mu\sin\theta+(1-\mu^2)\cos\theta\},$$

therefore $$T = Ce^{\mu\theta}+\frac{mga}{1+\mu^2}\{2\mu\sin\theta+(1-\mu^2)\cos\theta\}.$$

At the highest point $\theta = 0$ and $T = 0$; therefore $C = -mga\dfrac{1-\mu^2}{1+\mu^2}$.

Hence $$T = \frac{mga}{1+\mu^2}\{2\mu\sin\theta+(1-\mu^2)\cos\theta-(1-\mu^2)e^{\mu\theta}\}.$$

If the length of the string is that of a quadrant, we have $T = 0$ when $\theta = \dfrac{\pi}{2}$, and then μ is determined from the equation

$$e^{\frac{\mu\pi}{2}} = \frac{2\mu}{1-\mu^2}.$$

5. A, B, C are three unequally rough pegs in a vertical plane; P is the greatest mass that can be supported by a mass W when both are connected by a string (whose weight is neglected) passing over A, B, and C; Q is the greatest mass that W can support when the string passes over A and B; and R is the greatest that W can support when the string passes over B and C. Find the coefficients of friction for the pegs.

Employ a gravitation measure of force, and let the same symbol express the weight and the mass of a body, as explained in Art. 188. Let the inclinations of AB and BC to the vertical (measured in the same sense) be α and β, respectively; μ, μ', μ'' the coefficients of friction of A, B, C. Then, if the string passes over all the pegs and W hangs from A, it follows from equation (5) of Art. 196, that the tension, T, in the portion AB is $We^{\mu\alpha}$; and by the same equation, the tension, T', in BC is $Te^{\mu'(\beta-\alpha)}$; and, finally, $P = T'e^{\mu''(\pi-\beta)}$. Hence

$$P = We^{\mu\alpha+\mu'(\beta-\alpha)+\mu''(\pi-\beta)};$$

and the equations are obviously

$$\mu\alpha+\mu'(\beta-\alpha)+\mu''(\pi-\beta) = \log\frac{P}{W},$$

$$\mu\alpha+\mu'(\pi-\alpha) = \log\frac{Q}{W},$$

$$\mu'\beta+\mu''(\pi-\beta) = \log\frac{R}{W},$$

from which μ, μ', μ'' can be found. The value of μ' is $\dfrac{1}{\pi}\log\dfrac{QR}{PW}$.

6. A heavy uniform chain rests in limiting equilibrium on a rough

cycloidal arc, whose axis is vertical and vertex upwards, one extremity being at the vertex and the other at the cusp; prove that

$$e^{\frac{\mu\pi}{2}} = \frac{3}{1+\mu^2}.$$

(Wolstenholme's *Book of Math. Prob.*)

7. A uniform inextensible string whose length is l hangs in limiting equilibrium over a fixed rough cylinder of radius a whose axis is horizontal; find the lengths of the portions which hang vertically.

Ans. $\dfrac{l-\pi a}{1+e^{-\mu\pi}} + \dfrac{2\mu a}{1+\mu^2}$, and a value obtained by changing the sign of μ in this expression.

8. Two equal masses are attached each to the extremity of a string which hangs over a rough cylinder whose axis is horizontal; find how much either mass must be increased in order that it may begin to descend, the weight of the string being neglected.

Ans. The increase of mass $= P(e^{\mu\pi}-1)$, where P is the common value of the suspended masses.

9. A string, whose weight is neglected, passes *over* any number of equally rough fixed circular pulleys in a vertical plane; show that the ratio of two masses, suspended from the extremities of the string, which just sustain each other, is the same as if only one pulley were used.

10. A heavy uniform bar is moveable in a vertical plane round a smooth hinge at one extremity, and has the other extremity attached to a cord which passes over a small rough peg placed vertically over the hinge, and sustains a given weight; find the position of limiting equilibrium, and the tension of the cord.

Ans. If $W =$ weight of bar, $P =$ weight of suspended mass, T the tension, $2a =$ length of bar, $2c =$ distance of peg from hinge, $\theta =$ inclination of beam to vertical, and $\phi =$ inclination of cord to vertical, the position in which the bar is about to descend is given by the equations

$$c \sin\phi = a \sin(\theta - \phi),$$

$$T = Pe^{\mu(\pi-\phi)},$$

$$Wa \sin\theta = 2Tc \sin\phi.$$

11. A uniform inextensible string assumes the form of a circle under the influence of a repulsive force emanating from a point on its circumference; find the law of force.

Ans. It varies inversely as the cube of the distance.

12. A uniform inextensible string is in equilibrium under the action of a central repulsive force; prove that at each point of the string this force $\propto \dfrac{1}{p\gamma}$, where p is the perpendicular from the centre of force on the tangent, and γ the chord of curvature passing through the centre of force.

13. If the curve of equilibrium is an ellipse whose focus is the centre of force, the force at any point $\propto \frac{1}{rb'}$, where b' is the semi-conjugate diameter corresponding to the point, and r the focal distance of the point.

14. If the string assume the form of an ellipse under the influence of a repulsive force emanating from the centre, find the law of force.

Ans. The force is directly proportional to the distance, and inversely proportional to the conjugate diameter.

15. If an inextensible string can assume the same plane figure of equilibrium under the separate action of any number of forces, it can assume this figure under their combined action.

(To prove this, suppose the string under the combined action of the forces to be constrained to a smooth curve of the given figure, and it will follow that the pressure at every point of this curve varies inversely as the radius of curvature. The theorem follows, then, from example 2.)

16. A uniform inextensible string rests against the inner side of a smooth ellipse, and is repelled from the foci and the centre by the following forces: $\frac{\mu}{rb'}$ and $\frac{\mu'}{r'b'}$ emanating from the foci, and $\frac{\mu''a'}{b'}$ from the centre, the distances of a point on the string from the foci being r and r', respectively, its distance from the centre being a', and the semi-conjugate diameter corresponding to the point being b'. Find the pressure on the curve at any point.

Ans. If T_0 is the tension of the string at the extremity of the minor axis, R = pressure per unit length $= \frac{aT_0 - \mu - \mu' - \mu''a^2}{a\rho}$.

(The student will easily see from Examples 2 and 15, that if the curve of constraint of a string is a possible curve of free equilibrium under the action of the given forces, the pressure will, at every point, be $\frac{C}{\rho}$, where C is a constant. The result, in this example, might, therefore, be at once obtained by this principle.

By direct calculation, however, the result is obtained with little trouble. The equations of equilibrium are

$$dT + \frac{\mu}{rb'}\,dr + \frac{\mu'}{r'b'}\,dr' + \frac{\mu''a'}{b'}\,da' = 0,$$

$$\frac{T}{\rho} + R = \left(\frac{\mu}{r} + \frac{\mu'}{r'}\right)\frac{b}{b'^2} + \frac{\mu''ab}{b'^3};$$

and the first gives, by integration,

$$T - \frac{\mu}{a}\sqrt{\frac{r'}{r}} - \frac{\mu'}{a}\sqrt{\frac{r}{r'}} - \mu''b' = \text{const.})$$

The student will do well to apply the principle explained here to the kinetical examples in Walton, pp. 295 and 299, second edition.

cycloidal arc, whose axis is vertical and vertex upwards, one extremity being at the vertex and the other at the cusp; prove that

$$e^{\frac{\mu\pi}{2}} = \frac{3}{1+\mu^2}.$$

(Wolstenholme's *Book of Math. Prob.*)

7. A uniform inextensible string whose length is l hangs in limiting equilibrium over a fixed rough cylinder of radius a whose axis is horizontal; find the lengths of the portions which hang vertically.

Ans. $\dfrac{l-\pi a}{1+e^{-\mu\pi}} + \dfrac{2\mu a}{1+\mu^2}$, and a value obtained by changing the sign of μ in this expression.

8. Two equal masses are attached each to the extremity of a string which hangs over a rough cylinder whose axis is horizontal; find how much either mass must be increased in order that it may begin to descend, the weight of the string being neglected.

Ans. The increase of mass $= P(e^{\mu\pi}-1)$, where P is the common value of the suspended masses.

9. A string, whose weight is neglected, passes *over* any number of equally rough fixed circular pulleys in a vertical plane; show that the ratio of two masses, suspended from the extremities of the string, which just sustain each other, is the same as if only one pulley were used.

10. A heavy uniform bar is moveable in a vertical plane round a smooth hinge at one extremity, and has the other extremity attached to a cord which passes over a small rough peg placed vertically over the hinge, and sustains a given weight; find the position of limiting equilibrium, and the tension of the cord.

Ans. If $W =$ weight of bar, $P =$ weight of suspended mass, T the tension, $2a =$ length of bar, $2c =$ distance of peg from hinge, $\theta =$ inclination of beam to vertical, and $\phi =$ inclination of cord to vertical, the position in which the bar is about to descend is given by the equations

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11. A uniform inextensible string assumes the form of a circle under the influence of a repulsive force emanating from a point on its circumference; find the law of force.

Ans. It varies inversely as the cube of the distance.

12. A uniform inextensible string is in equilibrium under the action of a central repulsive force; prove that at each point of the string this force $\propto \dfrac{1}{p\gamma}$, where p is the perpendicular from the centre of force on the tangent, and γ the chord of curvature passing through the centre of force.

13. If the curve of equilibrium is an ellipse whose focus is the centre of force, the force at any point $\propto \frac{1}{rb'}$, where b' is the semi-conjugate diameter corresponding to the point, and r the focal distance of the point.

14. If the string assume the form of an ellipse under the influence of a repulsive force emanating from the centre, find the law of force.

Ans. The force is directly proportional to the distance, and inversely proportional to the conjugate diameter.

15. If an inextensible string can assume the same plane figure of equilibrium under the separate action of any number of forces, it can assume this figure under their combined action.

(To prove this, suppose the string under the combined action of the forces to be constrained to a smooth curve of the given figure, and it will follow that the pressure at every point of this curve varies inversely as the radius of curvature. The theorem follows, then, from example 2.)

16. A uniform inextensible string rests against the inner side of a smooth ellipse, and is repelled from the foci and the centre by the following forces: $\frac{\mu}{rb'}$ and $\frac{\mu'}{r'b'}$, emanating from the foci, and $\frac{\mu''a'}{b'}$ from the centre, the distances of a point on the string from the foci being r and r', respectively, its distance from the centre being a', and the semi-conjugate diameter corresponding to the point being b'. Find the pressure on the curve at any point.

Ans. If T_0 is the tension of the string at the extremity of the minor axis, R = pressure per unit length $= \frac{aT_0 - \mu - \mu' - \mu''a^2}{a\rho}$.

(The student will easily see from Examples 2 and 15, that if the curve of constraint of a string is a possible curve of free equilibrium under the action of the given forces, the pressure will, at every point, be $\frac{C}{\rho}$, where C is a constant. The result, in this example, might, therefore, be at once obtained by this principle.

By direct calculation, however, the result is obtained with little trouble. The equations of equilibrium are

$$dT + \frac{\mu}{rb'}\,dr + \frac{\mu'}{r'b'}\,dr' + \frac{\mu''a'}{b'}\,da' = 0,$$

$$\frac{T}{\rho} + R = \left(\frac{\mu}{r} + \frac{\mu'}{r'}\right)\frac{b}{b'^2} + \frac{\mu''ab}{b'^2};$$

and the first gives, by integration,

$$T - \frac{\mu}{a}\sqrt{\frac{r'}{r}} - \frac{\mu'}{a}\sqrt{\frac{r}{r'}} - \mu''b' = \text{const.})$$

The student will do well to apply the principle explained here to the kinetical examples in Walton, pp. 295 and 299, second edition.

SECTION II.

Flexible Extensible Strings.

197.] **Experimental Law of Extension.** The strings which we now proceed to consider are *extensible,* i.e. such as have their lengths increased when they are in a state of tension. For such strings we shall still assume the property of complete flexibility as defined in Art. 181.

The law of extension which we proceed to enunciate applies not only to flexible strings but also to straight bars of iron, steel, &c.

Let l_0 denote the length of any string or straight bar of uniform section when it is not subject to the action of any external force. This is called the *natural length* of the string or bar. Let σ be the area of the normal section, F the magnitude of the force applied at one extremity in the direction AB, of the string or bar. Then, supposing the extremity A to be fixed, the force F will produce an extension, BC, of the body. Denote this extension by x. Then experiment proves that *for small values* of the ratio $\frac{x}{l_0}$ in the case of solid bars there is for the same bar a constant ratio between this fraction and the quantity $\frac{F}{\sigma}$; and there is the same constancy of ratio in the case of strings, but for some of these latter bodies the value of $\frac{x}{l_0}$ may be very much greater than for bars.

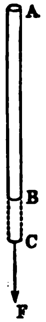


Fig. 237.

We have, then,

$$\frac{F}{\sigma} = E\frac{x}{l_0}, \tag{1}$$

E being a constant quantity which is called *Young's modulus of elasticity* of the matter of which the string or bar is formed. Since $\frac{x}{l_0}$ is a number, it follows that E is *a force per unit of sec-*

tional area, and it is evidently a measure of the longitudinal rigidity of the substance.

If the law expressed by equation (1) be supposed to hold for an extension x equal to l_0, and if the force applied to the body to produce this extension be called P, we have $E = \frac{P}{\sigma}$; and if σ is a section of unit area, $E = P$. Young's modulus of elasticity of any substance might then be defined as that force which, if applied at the extremity of a bar of the material of unit section, would double its length—this force being fictitious in the case of bars or strings for which (1) holds only within extremely narrow limits.

For bars of iron and steel this equation is true only within narrow limits—called *the limits of elasticity*—while for flexible strings of such substances as India-rubber its range is much wider. If the limiting amount of extension has not been surpassed, the body will, after a time varying with the substance, return to its original state when the stretching force F is removed. The law expressed by equation (1) is also true within narrow limits in the case of a straight bar which is *compressed* without bending.

A notion of the magnitude of the modulus of elasticity of a solid body may be formed from the fact that in the case of iron, the unit of force being a kilogramme weight and the unit of area a square centimetre, E is about 2,000,000. For what are commonly called elastic strings, E is of course very much smaller than for bars of iron or steel.

In the case of an elastic string it is usual to put equation (1) into another form. If l is the length which the string assumes under a tension T, we have $x = l - l_0$, and

$$\frac{T}{\sigma} = E\frac{l-l_0}{l_0},$$

or

$$l = l_0\left(1 + \frac{T}{E\sigma}\right).$$

or, as it is usually written,

$$l = l_0\left(1 + \frac{T}{\lambda}\right), \tag{2}$$

the quantity λ being called the modulus of elasticity of the string.

This quantity is obviously the force which must be applied to the string to double its length.

The law expressed by (1) or (2) is known as *Hooke's Law*, from the name of its discoverer, and is sometimes expressed in the form—*the tension of any elastic string is proportional to its extension beyond its natural length.*

198.] **Work done in slowly extending a String or Bar.** If at each instant during the extension of a string or bar the stretching force applied at the extremity is exactly equal to that which would keep the body in its state of deformation at this instant, there is continuous equilibrium between the (gradually increasing) applied force and the elastic force of the body, and therefore the total amount of work done by the applied force is equal to the work done against the internal force.

If the bar or string is not slowly extended, its tension will not at each instant be equal to the applied force; but it is still true that its tension and extension are connected by equation (2), so that the expression for work done which we are about to investigate will be that which, with a negative sign prefixed, gives the work done by the varying tension. The work of the external force will be numerically greater, the excess producing kinetic energy.

Now if x is the extension of the body at any instant, the corresponding force is $\frac{E\sigma}{l_0}x$, and the work done by this force in a further extension dx is $\frac{E\sigma}{l_0}x\,dx$. Let a be the final extension; then the total work done is

$$\int_0^a \frac{E\sigma}{l_0}x\,dx, \quad \text{or} \quad \frac{E\sigma a^2}{2l_0},$$

the extension being, of course, confined within the limits of elasticity. The applied force which is required to keep the body in its final state of extension is, by (1) of last Article, $\frac{E\sigma a}{l_0}$. Hence if the force applied in the final state be denoted by P, the whole amount of work done is

$$\tfrac{1}{2}Pa,$$

or half the work which would be done by the *final* force of extension in moving its point of application through a space equal to the final extension.

199.] Equations of Equilibrium of an Extensible String. Suppose the string to have assumed its figure of equilibrium under the action of the given forces. At any point of the string let ds be the stretched length of an element whose natural length was ds_0; and at this point let m be the mass per unit length, the mass per unit length at the same point in the natural state of the string being m_0.

Then, since the mass of the element ds is the same after as before stretching,

$$m\,ds = m_0\,ds_0. \tag{1}$$

Also by Hooke's law

$$ds = \left(1 + \frac{T}{\lambda}\right) ds_0. \tag{2}$$

But, the string having assumed its form of equilibrium, we have, as for the inextensible string,

$$\left.\begin{aligned} \frac{d}{ds}\left(T\frac{dx}{ds}\right) + mX = 0, \\ \frac{d}{ds}\left(T\frac{dy}{ds}\right) + mY = 0. \end{aligned}\right\} \tag{3}$$

Also

$$ds = \sqrt{dx^2 + dy^2}; \tag{4}$$

and since the nature of the string in its original state is supposed to be specified, we shall have m_0 given as a function of the position of the element ds_0 in the natural state; or

$$m_0 = f(s_0), \tag{5}$$

where s_0 is the length of the arc of the original string measured from some origin point on it.

Now the general problem of extensible strings may be stated as follows:—*An extensible string, the law of variation of whose density in its natural state is given, is, under given circumstances, submitted to the action of given forces; find the form which it will assume.*

To solve this problem, it is necessary to find an equation between x and y, the co-ordinates of any point in the stretched string; and as the equations just given contain, in addition to these co-ordinates, the quantities m, m_0, s, s_0, and T, these must be eliminated. But from the six equations above, these five quantities may theoretically be eliminated, by differentiation or otherwise, and there will result a single equation between x and y, which is that of the curve of equilibrium.

The problem in its general form is one which it would often be practically impossible to solve. We shall therefore in the sequel consider only two cases—viz. that in which m_0 is the same throughout the string, and that in which the external forces are constant. Consider m_0 constant.

Multiplying the left-hand sides of equations (3) by $\frac{dx}{ds}$ and $\frac{dy}{ds}$, and adding,

$$\frac{dT}{ds} + m\left(X\frac{dx}{ds} + Y\frac{dy}{ds}\right) = 0\,; \tag{6}$$

while from (1) and (2) we have

$$m = \frac{m_0}{1 + \frac{T}{\lambda}}.$$

Hence (6) becomes

$$\left(1 + \frac{T}{\lambda}\right)dT + m_0(Xdx + Ydy) = 0. \tag{7}$$

Integrating,

$$\frac{\lambda}{2}\left(1 + \frac{T}{\lambda}\right)^2 + m_0\int(Xdx + Ydy) = \text{const.} = A.$$

Denoting, as in Art. 195, the general integral by V,

$$\frac{\lambda}{2}\left(1 + \frac{T}{\lambda}\right)^2 = A - V, \tag{8}$$

or by (2),

$$\frac{ds}{\sqrt{A - V}} = \sqrt{\frac{2}{\lambda}} \cdot ds_0, \tag{9}$$

from which the relation between s and s_0 is found, and hence the extension of the string. Equation (8) is the analogue of that of Art. 195. If V' is the potential of the external forces at a point at which the tension is T', we have

$$(T - T')\left(1 + \frac{T + T'}{2\lambda}\right) = V' - V. \tag{10}$$

The equation of the curve of equilibrium is obtained by substituting the value of T given by (8) in either of the equations (3)—suppose in the equation

$$\left(1 + \frac{T}{\lambda}\right)\frac{d}{ds}\left(T\frac{dx}{ds}\right) + m_0X = 0.$$

Secondly, suppose that the applied forces X, Y are constant.

Then from (3) $$T\frac{dx}{ds} = A - X\int m_0 ds_0, \tag{11}$$

A being a constant of integration. Similarly

$$T\frac{dy}{ds} = B - Y\int m_0 ds_0. \tag{12}$$

Hence $$T^2 = (A - X\int m_0 ds_0)^2 + (B - Y\int m_0 ds_0)^2, \tag{13}$$

which gives T as a function of s_0, i.e. the tension at any point in the stretched string in terms of the length of the arc of the unstretched string measured up to the corresponding point. In other words, $$T = \phi(s_0). \tag{14}$$

Therefore from (2)

$$s = \int\left\{1 + \frac{\phi(s_0)}{\lambda}\right\}ds_0,$$

which gives the relation between the stretched and unstretched lengths of any portion. The equation of the curve of equilibrium is obtained from (11) and (12) thus:

$$dx = (A - X\int m_0 ds_0)\frac{ds}{T}$$

$$= (A - X\int m_0 ds_0)\left\{\frac{1}{\lambda} + \frac{1}{\phi(s_0)}\right\}ds_0.$$

Similarly $$dy = (B - Y\int m_0 ds_0)\left\{\frac{1}{\lambda} + \frac{1}{\phi(s_0)}\right\}ds_0.$$

Integrating these equations and eliminating s_0 between them, we obtain the equation of the curve.

As an example, let it be proposed to investigate the form of an elastic string suspended from two fixed points and acted on by gravity, the string being uniform in its natural state. Taking axes as in Art. 184, we have,

$$\frac{d}{ds}\left(T\frac{dx}{ds}\right) = 0,$$

$$\frac{d}{ds}\left(T\frac{dy}{ds}\right) = mg.$$

Hence $T\frac{dx}{ds} = \tau = m_0 gc$, suppose; and $T\frac{dy}{ds} = B + m_0 g s_0$. But if s_0 be measured from the lowest point, $\frac{dy}{ds} = 0$ and $s_0 = 0$ at the same time. Hence $B = 0$, and we have

$$T\frac{dx}{ds} = m_0 gc,$$

$$T\frac{dy}{ds} = m_0 g s_0;$$

from which $T = m_0 g\sqrt{c^2+s_0^2}$; therefore

$$dx = \left(\frac{m_0 gc}{\lambda} + \frac{c}{\sqrt{c^2+s_0^2}}\right) ds_0,$$

$$dy = \left(\frac{m_0 g s_0}{\lambda} + \frac{s_0}{\sqrt{c^2+s_0^2}}\right) ds_0.$$

Hence, putting $\lambda = m_0 ga$, we have

$$x = \frac{cs_0}{a} + c\log\frac{s_0+\sqrt{c^2+s_0^2}}{c}, \tag{15}$$

$$y = \frac{s_0^2}{2a} + \sqrt{c^2+s_0^2}. \tag{16}$$

In the hyperbolic notation, put $s_0 = c\sinh\omega$, Art. 184; then we have

$$x = c\omega + \frac{c^2}{a}\sinh\omega, \tag{α}$$

$$y = c\cosh\omega + \frac{c^2}{2a}\sinh^2\omega$$

$$= c\cosh\omega + \frac{c^2}{2a}\cosh^2\omega - \frac{c^2}{2a},$$

and by taking the axis of x at a distance $\frac{c^2}{2a}$ below the lowest point of the curve, this last equation becomes

$$y = c\cosh\omega + \frac{c^2}{2a}\cosh^2\omega. \tag{β}$$

The relation between x and y is obtained by eliminating s_0 or ω from these equations.

An approximate relation between them may be obtained when the string is only slightly extensible, i.e., when λ (or a) is very great. In this case (16) gives

$$s_0^2 = (y^2-c^2)\left(1 - \frac{y}{a} + \frac{5y^2-c^2}{a^2}\right), \tag{17}$$

to the second order of the small quantity $\frac{1}{a}$.

Now, writing (15) and (16) in the forms

$$x = \frac{cs_0}{a} + \xi, \quad y = \frac{s_0^2}{2a} + \eta,$$

we know that $\eta = \frac{c}{2}\left(e^{\frac{\xi}{c}} + e^{-\frac{\xi}{c}}\right).$

Hence
$$y - \frac{s_0^2}{2a} = \frac{c}{2}\left(e^{\frac{x}{c}} . e^{-\frac{s_0}{a}} + e^{-\frac{x}{c}} . e^{\frac{s_0}{a}}\right)$$
$$= u - \frac{2vs_0}{a} + \frac{us_0^2}{2a^2},$$
by expanding $e^{\frac{s_0}{a}}$ and $e^{-\frac{s_0}{a}}$ as far as $\frac{1}{a^2}$ and denoting by u and v the quantities $c \cosh \frac{x}{c}$ and $c \sinh \frac{x}{c}$.

Substituting in this equation the value of s_0 given by (17)—in which it is evident that the term of the second order may be rejected if we desire to obtain y to this order only in terms of x—we obtain an equation of the form
$$y = u + \frac{P}{a} + \frac{Q}{a^2}, \tag{18}$$
in which P and Q are both functions of x and y.

Now assume $y = u + \frac{\lambda}{a} + \frac{\mu}{a^2}$, where λ and μ are functions of x alone, and substitute this value of y in every term of (18). This will give us, with a little trouble,
$$\lambda = -\frac{1}{2}v^2, \text{ and } \mu = \frac{1}{2}uv^2.$$

Hence, finally, $$y = u - \frac{v^2}{2a} + \frac{vu^2}{2a^2},$$
to the second order of the small quantity $\frac{1}{a}$.

200.] Extensible String on Smooth Curve. It is clear that the equations (1) and (2) of Art. 193 are applicable to an extensible string, as are also those of Article 196 for a rough curve. The result arrived at by integration, which expresses the tension in terms of the potential, is to be replaced by equation (10) of Art. 199; and from this equation it follows that if an extensible string, uniform in its natural state, rest on any smooth surface under the action of gravity, the free extremities are in the same horizontal plane.

EXAMPLES.

1. An elastic string, uniform in its natural state, is suspended from one extremity, which is fixed, and has a given mass attached to the other; find the extension of the string, taking its own weight into account.

Let W be the weight of the string, P the weight of the suspended mass, λ the modulus of elasticity, and m_0 the mass of a unit length of the unstretched string. Then the equation of equilibrium is

$$dT + m_0 g ds_0 = 0.$$

If l_0 is the natural length of the string, $m_0 g l_0 = W$; therefore this equation gives by integration

$$T + \frac{W}{l_0} s_0 = \text{const.}$$

When $s_0 = 0$, T is evidently $W + P$; therefore

$$T = W + P - \frac{W}{l_0} s_0.$$

Again, since $ds = (1 + \frac{T}{\lambda}) ds_0$, we have

$$ds = (1 + \frac{W+P}{\lambda} - \frac{W}{\lambda l_0} s_0) ds_0,$$

$$\therefore \quad s = (1 + \frac{W+P}{\lambda} - \frac{W}{2\lambda l_0} s_0) s_0,$$

no constant being added because $s = 0$ when $s_0 = 0$.

If $s_0 = l_0$, and l is the whole length of the stretched string, we have

$$l = l_0 (1 + \frac{W + 2P}{2\lambda}).$$

2. A heavy uniform elastic ring is placed round a smooth vertical cone; find how far it will descend.

Let W be the weight of the ring, $2\pi a$ its natural length, λ its modulus of elasticity, y the distance of the plane of the ring from the vertex of the cone in the position of equilibrium, and l the stretched length in this position. Then if the ring be moved down through an indefinitely small vertical distance, δy, the equation of work is

$$-T\delta l + W\delta y = 0,$$

T being the tension of the ring. If a is the semi-vertical angle of the cone, $l = 2\pi y \tan a$; hence $\delta l = 2\pi \tan a \,.\, \delta y$, and

$$2\pi T \tan a = W.$$

But, by Hooke's Law,

$$y \tan a = a (1 + \frac{T}{\lambda});$$

$$\therefore \quad y = a \cot a (1 + \frac{W}{2\pi\lambda} \cot a).$$

3. An elastic string, uniform in its original state, is placed on any smooth curve and acted on by given forces; find its extension.

The tension at any point is determined by the equation

$$(1 + \frac{T}{\lambda}) dT + m_0 (X dx + Y dy) = 0,$$

or

$$\lambda (1 + \frac{T}{\lambda})^2 + 2m_0 \int (X dx + Y dy) = \text{const.} \tag{1}$$

Let $m_0 \int (X dx + Y dy)$ be denoted by V. Now take any point, O, in the string as the point from which s and s_0 are measured, and let A be the value of V at a free extremity of the string. If one extremity is fixed, it will be well to measure s and s_0 from it. Putting

$T = 0$, $V = A$, and also $1 + \frac{T}{\lambda} = \frac{ds}{ds_0}$,

(1) gives $$\left(\frac{ds}{ds_0}\right)^2 = 1 + \frac{2m_0}{\lambda}(A - V). \tag{2}$$

Suppose the curve of constraint to be given by the two equations

$$x = f_1(s), \quad y = f_2(s).$$

Then (2) gives $$\frac{ds}{\sqrt{1 + \frac{2m_0}{\lambda}(A - V)}} = ds_0,$$

or, by integration, $$\phi(s, A) = s_0 + \phi(0, A), \tag{3}$$

s and s_0 being both measured from O. Let l and l_0 be the stretched and original lengths of the portion between O and the free extremity considered. Then we have

$$\phi(l, A) = l_0 + \phi(0, A). \tag{4}$$

But A is evidently a function of the co-ordinates of the extremity, and these co-ordinates are, by supposition, $f_1(l), f_2(l)$; hence A is a known function of l, and by substituting its value in (4) we deduce the value of l.

4. One extremity of an elastic string, originally uniform, is fixed at the highest point of a smooth cycloid in a vertical plane, the string lying along the convex side of the curve; find the extension produced by gravity.

If the tangent at the highest point is taken as axis of x, and if $\frac{\lambda}{2m_0 g}$ is denoted by c, we find easily, for any curve of constraint,

$$\frac{ds}{\sqrt{c + h - y}} = \frac{ds_0}{\sqrt{c}},$$

h being the ordinate of the free extremity.

In the cycloid $s^2 = 8ay$. Substituting this value of y in the equation, and integrating, we have

$$s = 2\sqrt{2a(c+h)} \sin\left(\frac{s_0}{2\sqrt{2ac}}\right).$$

If l be the length from the fixed to the free extremity, and l_0 the natural length of the string,

$$l = 2\sqrt{2a(c+h)} \sin\left(\frac{l_0}{2\sqrt{2ac}}\right).$$

Also $$l^2 = 8ah.$$

These equations combined give

$$l = 2\sqrt{2ac}\tan\left(\frac{l_0}{2\sqrt{2ac}}\right).$$

5. A heavy particle is attached to one end of an elastic string whose unstretched length is indefinitely small; the particle rests on a smooth curve in a vertical plane, and the fixed end of the string is attached to a point in this curve; find the nature of the curve so that the particle may rest in all positions. *Ans.* A cycloid.

6. A heavy elastic string is laid upon a smooth double inclined plane in such a manner as to remain at rest; find how much the string is stretched. (Walton, p. 140.)

Ans. If W is the weight, λ the modulus of elasticity, and c the natural length of the string, and a, a' the inclinations of the planes to the horizon, the extension is

$$\frac{W}{2\lambda}\,\frac{\sin a\sin a'}{\sin a+\sin a'}\,c.$$

[For the portion on the plane a let s and s_0 be measured from the free extremity. Then

$$T=\frac{W\sin a}{c}s_0;\text{ and } ds=\left(1+\frac{T}{\lambda}\right)ds_0=\left(1+\frac{W\sin a}{\lambda c}s_0\right)ds_0.$$

Hence if l is the length of the portion on the plane a, we have

$$l=l_0\left(1+\frac{W\sin a}{2\lambda c}l_0\right).$$

A similar equation holds for the portion on the plane a'. Now the extension $= l+l'-l_0-l'_0$; and equating the tensions at the common summit of the planes, we have $l_0\sin a = l'_0\sin a'$,

$$\therefore\quad l_0=\frac{c\sin a'}{\sin a+\sin a'},\ \&\text{c.}]$$

7. If the cone in example 2 is replaced by a smooth paraboloid of revolution, find how far the ring will descend. [By Virtual Work.]

Ans. $y=\dfrac{a}{1-\dfrac{aW}{4\pi m\lambda}}$, where $4m$ = latus rectum of generating parabola.

8. An elastic string, uniform in its original state, rests on a rough inclined plane with its upper extremity fixed; prove that its extension will lie between the limits

$$\frac{l^2}{2c}\cdot\frac{\sin(i\pm\epsilon)}{\cos\epsilon},$$

where i = inclination of plane, ϵ = angle of friction, l = natural length of string, and c = length of a portion of the string in its natural state whose weight is the modulus of elasticity. (Wolstenholme's *Math. Prob*)

9. A mass of weight P just supports another of weight Q by means of a fine elastic string passing over a rough circular cylinder whose axis is horizontal; λ is the modulus of elasticity, and a the radius of the cylinder; prove that the extension of the part of the string in contact with the cylinder is

$$\frac{a}{\mu} \log \frac{Q+\lambda}{P+\lambda}.$$ (Wolstenholme, *ibid.*)

10. Two uniform ladders, connected by a smooth axis at a common extremity, rest in a vertical plane with their other extremities, which are connected by an elastic rope, on a rough horizontal plane; find the greatest angle between them consistent with equilibrium.

Ans. If a is the length of each ladder, $2a \sin a$ the natural length of the rope, 2θ the greatest angle between the ladders, and λ the modulus of elasticity of the rope,

$$\lambda (\sin \theta - \sin a) = W \sin a \, (\mu + \tfrac{1}{2} \tan \theta).$$

11. A heavy uniform elastic ring is placed horizontally round a right cone whose axis is vertical and vertex upwards, the stretched ring being also uniform; find its extreme positions of equilibrium.

Ans. $y = a \cot a \left\{1 + \dfrac{W}{2\pi\lambda} \cot (a \pm \epsilon)\right\}$, with notation of Ex. 2.

12. A heavy elastic string, uniform in its natural state, is placed round a smooth fixed circular cylinder whose axis is horizontal, and is just out of contact with the lowest point of the cylinder; determine the tension at any point.

Ans. Let r = radius of cylinder, p = weight per unit length of string in its natural state, λ = modulus of elasticity, and θ = inclination of any radius to the vertical; then the tension at the end of this radius is given by the equation

$$\left(1 + \frac{T}{\lambda}\right)^2 = \frac{2rp}{\lambda} \cos \theta + \frac{\lambda + 6rp + \sqrt{\lambda^2 + 4\lambda rp}}{2\lambda}.$$

13. A naturally uniform heavy elastic string is hung up by one end. Prove that if the modulus of elasticity is equal to half the weight of the string, the string will stretch to double its natural length.

If a uniform string fixed at one end be acted on by a force in the direction of its length, find the law of force in order that the tension at any point may vary as the square of the distance from the free end. (College Examinations, Cambridge, 1887.)

Ans. At a distance s from the fixed end the force per unit mass is

$$2\mu (l-s) \left\{1 + \frac{\mu}{\lambda} (l-s)^2\right\},$$

where l is the stretched length of the string, viz.,

$$\sqrt{\frac{\lambda}{\mu}} \tan \left(a \sqrt{\frac{\mu}{\lambda}}\right),$$

a being the natural length, and the tension at any point is represented by $\mu (l-s)^2$.

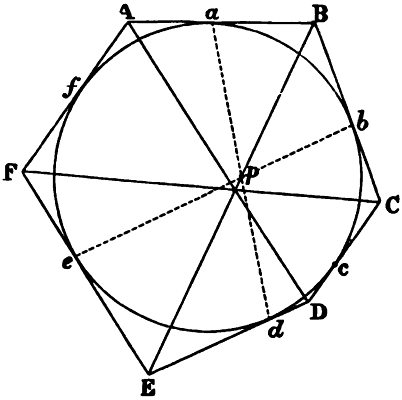
APPENDIX.

A.

STATICAL PROOF OF BRIANCHON'S THEOREM.

For the following statical proof of this theorem, which supplies the deficiency in Art. 100, I am indebted to Mr. E. M. Langley of Bedford.

Let $ABCDEF$ be any hexagon circumscribed to a circle. Now ad and be being any two chords of a circle, if we have forces represented by $\frac{k}{ad}$ and $\frac{k}{be}$, acting along ad and be, respectively, their resultant acts in the line BE, which is obtained by joining the point of intersection of the tangents at a, b to the point of intersection of those at b, e. This follows simply from the fact (see example 12, Art. 22) that the resultant of $\frac{k}{ad}$ and $\frac{k}{ba}$ acts along aB, and the resultant of $\frac{k}{ab}$ and $\frac{k}{be}$ acts along bB; but the forces $\frac{k}{ad}$, $\frac{k}{be}$ are equivalent to the four forces $\frac{k}{ad}$, $\frac{k}{ab}$, $\frac{k}{ba}$, $\frac{k}{be}$, since the middle pair destroy each other; hence the resultant of $\frac{k}{ad}$ and $\frac{k}{be}$ acts through B. Similarly it acts, of course, through E; and therefore its line of action is BE. [It acts also through p, $\therefore$ BpE is a right line.]



Now take the null system of forces

$$\frac{k}{ad}, \frac{k}{be}; \quad \frac{k}{cf}, \frac{k}{da}; \quad \frac{k}{eb}, \frac{k}{fc}.$$

The resultant of the first pair acts along BE, that of the second pair acts along DA, and that of the last along FC; and since the resultant of the whole set vanishes, the lines

BE, CF, and DA

are concurrent. This is Brianchon's Theorem.

B.

EXAMPLES, CHIEFLY NUMERICAL.

1. $ABCD$ is a rectangle, of which the side $AD = 8$ inches and $AB = 15$; find the line of action of the resultant of the following forces:

22 from D to A, 10 from A to B, 51 from D to B, 34 from A to C, 40 from C to B, and 35 from C to D.

Result. If AB and AD are taken as axes of x and y, the equation of the resultant is

$$7x + 5y - 68 = 0.$$

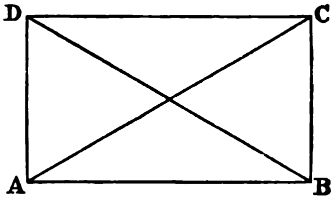


Fig. a.

2. If all the forces in the preceding remain unaltered except that 35 in CD is replaced by 15 in the same sense, find the line of action of the resultant.

Its equation is $x + y - 12 = 0$.

3. If the force 34 in Q. 1, instead of acting in AC, acts at B parallel to AC, find the line of action of the resultant.

Its equation referred to the same axes is

$$7x + 5y - 44 = 0.$$

4. If in Q. 1, the force 34 is transferred to B and the force 51 to A, each with unaltered direction, find the line of action of the resultant.

Its equation is $7x + 5y - 8 = 0$.

5. $ABCDEF$ is a regular hexagon; find the resultant of the following forces (each measured in pounds' weight, suppose):

20 from A to B, 6 from C to B, 14 from D to C, 16 from A to D, 25 from D to E, 30 from F to E, 2 from F to A.

Result. $R = 31$ pounds' weight; its components parallel to AB and AE are 23 and $12\sqrt{3}$, and the perpendicular on its line of action from the middle point of AD is $\frac{3}{31}p$, where $p =$ the perpendicular from this point on a side of the hexagon.

6. ABC is a triangle in which $AB = 14$ inches, $AC = 15$, $BC = 13$, and CP is perpendicular to AB; find the resultant of the following forces (in pounds' weight, suppose):

40 from A to B, 26 from B to C, 30 from C to A, 35 from P to C.

Result. $R = 37$, and its line of action has for equation, referred to PB and PC as axes of x and y,

$$35x - 12y - 236 = 0.$$

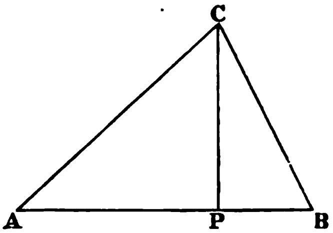


Fig. β.

7. If in fig. β, $AB = 28$ inches, $BC = 17$, $AC = 25$, find the resultant of the following forces:

28 from B to A, 35 from A to C, 51 from B to C, 34 from C to P.

Result. $R = 40$, and the equation of its line of action referred to the same axes is $8x - 6y - 15 = 0$.

8. If in the last the force acting from B to A is 18, while in addition to the other given forces there is a force of 10 from the middle point of BC to the middle point of CA find the line of action of the resultant.

Result. $32x - 24y + 15 = 0$.

9. $ABCD$ is a trapezium in which $AB = 40$ inches, $BC = 17$, $CD = 12$, $DA = 25$, and the perpendicular distance between AB and $CD = 15$; find the resultant of the following forces:

38 from B to A, 50 from A to D, 40 from C to D, 68 from B to C, 70 from D to B.

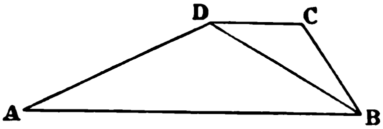


Fig. γ.

Result. $R = 50$, and the equation of its line of action referred to BA and the perpendicular to BA at B as axes of x and y is

$$24x - 7y - 300 = 0.$$

10. If the force 36 in BA is transferred from this line to that which joins the middle points of AD and BC, find the line of action of the resultant.

Referred to the same axes its equation is

$$24x - 7y - 165 = 0.$$

11. A, B, C are three points such that $AB = 14$ inches, $BC = 13$, $CA = 15$; the sum of the moments round A of a system of forces in the plane ABC is 900 inch-pounds' weight, counterclockwise; about B 500, clockwise; about C 2880, counterclockwise; find the magnitude and line of action of the resultant force.

Magnitude = 260 pounds' weight; line of action cuts AB at a distance 5 from B and makes $\tan^{-1}\frac{5}{12}$ with AB.

12. *AB* is a bar freely moveable in a vertical plane round a horizontal axis fixed through the end *A*; to the end *B* is attached a cord which, passing over a pulley at *C*, fixed vertically above *A*, sustains a mass of weight *P* from its free extremity. If *W* is the weight of the bar, and if its centre of gravity, *G*, divides *AB* so that $AG = \frac{1}{n} AB$, show that in the position of equilibrium the point *B* is determined by the intersection of a circle of radius *AB* about *A* as centre with a circle round *C* as centre, the radius of this latter being

$$n\frac{P}{W} \cdot CA.$$

13. *AC* and *CB* are two uniform bars of equal lengths but of weights *P* and *Q*, respectively; they are in the same vertical plane and are freely jointed together at *C*, while the first is moveable round a smooth horizontal axis fixed through *A*, and the end *B* slides along a rough horizontal plane passing through *A*; find the limiting position of equilibrium.

Result. If θ is the limiting inclination of *AC* to the horizon and μ the coefficient of friction between *B* and the ground,

$$\tan\theta = \frac{P+Q}{\mu(P+3Q)}.$$

14. Solve the preceding problem when the bars have different lengths as well as different weights.

Result. If θ and ϕ are the values of the angles at *A* and *B*, $AC = 2a$, $BC = 2b$, we have

$$a\sin\theta = b\sin\phi, \qquad (1)$$

$$Q\tan\theta + (P+2Q)\tan\phi = \frac{P+Q}{\mu}. \qquad (2)$$

15. Show how to determine θ and ϕ from these equations by graphic construction.

Draw a right line *LO* to represent *Q* on any scale; produce *LO* through *O* to *M* so that *OM* represents $P+2Q$; at *O* draw *OD* perpendicular to *LM* and let *OD* represent $\frac{P+Q}{2\mu}$; take on *OD* two points, *m*, *n*, equidistant from *D* on opposite sides; then the lines *Lm* and *Mn* intersect in a point *P*, which (by varying *m* and *n*) traces out a curve which is the di-polar locus representing all angles satisfying the equation (2); i.e. *P* being any point on this curve, *PLM* and *PMN* will be values of θ and ϕ, respectively.

The di-polar locus representing (1) is, of course, a circle; for if $\frac{PM}{PL} = \frac{b}{a}$, the problem is: given the base, *LM*, of a triangle and the ratio of its sides, find the locus of its vertex, *P*. The locus is a circle whose diameter is *XY*, where *X* is the point between *L* and *M* such that $\frac{LX}{XY}=\frac{a}{b}$, and Y is the point dividing LM externally in the same ratio.

The point of intersection of these two simply constructed loci is the point which determines the values of θ and ϕ satisfying both (1) and (2).

16. *ABC* represents a solid triangular prism resting with its base *AB* on a rough inclined plane whose inclination is to be gradually increased, *A* being the lower and *B* the upper end of the base. Given the perpendicular height of the prism (perpendicular from *C* on *AB*) and the base, but not the angles *A*, *B*, find the greatest value of the angle *A* so that the prism will begin to slide and not to tumble.

Result. If $AB = c$ and h is the height of the prism, while μ is the coefficient of friction, the greatest value of A is given by the equation

$$\cot A = \mu - \frac{c}{h}.$$

17. The frustum of a solid right cone rests on a rough inclined plane whose inclination is to be gradually increased; the diameter of the base on the plane is 8 feet, and the diameter of the upper face is 4 feet; if the coefficient of friction is $\frac{1}{2}$, what is the greatest height of the frustum that will ensure an initial motion of slipping?

Ans. $20\frac{4}{11}$ feet.

18. *AC* represents the floor of a bridge moveable round a fixed horizontal axis at *A*; to the extremity *C* is attached a chain of negligible weight which, passing over a pulley *D* fixed vertically over *A*, is attached to a mass of weight *P* at its second extremity; this mass *P* is constrained to move along a fixed curve in a vertical plane, this curve being such that the bridge is in equilibrium in all positions.

Show how to construct this curve. [Theory of a *Bascule Bridge.*]

Solution. Let the horizontal position of *AC* be *AB*, and in this position let *P* be at P_0 (which may coincide with *D*); find the centre of gravity of P_0 and the floor *AB*, and draw through this point a horizontal line *LM*. Now, frictional resistances being neglected, the centre of gravity of *P* and *AC* must move along this line *LM* in all positions of the system, in order that all positions should be positions of equilibrium. Let *G* be the centre of gravity of *AC*, and *p* the perpendicular from *G* on *LM*; then if *W* is the weight of *AC*, the counterpoise, *P*, must be at a distance $\frac{W}{P}\cdot p$ from *LM*; but *P* lies also on a circle of given radius with *D* as centre (the radius being the difference between the whole length of the chain and the length *DC*); hence the position of *P* is found for every position of *AC*.

The curve locus of *P* is easily found to be one of the fourth degree.

19. The vertical rectangle *ABCD* represents a solid uniform

rectangular block whose base AB rests on the ground; LM is a bar whose extremity L is fixed on the ground adjacent to A, and whose extremity M rests against the face AD of the block. A gradually increasing load is applied at the middle point of LM, find the nature of the initial motion of the block, being given all necessary particulars.

Result. If h is the height AM, $2b = AB$, $\mu =$ coefficient of friction between the block and the ground, $\mu' =$ that at M, the block will begin to turn round B if

$$h > (\mu' + \frac{1}{\mu})\, b.$$

20. If in the last example $\mu = \frac{1}{4}$, $\mu' = \frac{1}{3}$, $AB = 8$ feet, $AM = 12$, $LA = 16$, $W =$ weight of block, find the value of the load when equilibrium ceases.

Result. Load $= \frac{13}{22}\, W$.

EXAMPLES OF CATENARIES.

21. Show that all common catenaries are similar curves.

Let there be two common catenaries with parameters c, c'; place them so that their directrices coincide, as also their axes. Then any line, OPP', drawn through their common origin O will meet them in points, P, P', such that $\frac{OP}{OP'} = \frac{c}{c'}$. This is, of course, directly obvious from the equation of any catenary, $\frac{y}{c} = \cosh\frac{x}{c}$; because if, retaining the same origin, we alter c to nc, we get a new curve in which x and y are each n times what they were in the old curve.

Hence all catenaries can be traced by simple multiplication or division from the curve $y = \frac{1}{2}(e^x + e^{-x})$.

This is an *a priori* necessity for every curve whose equation involves only one linear constant, e. g. parabolas, catenaries of uniform strength, &c.

22. A uniform chain hangs over two fixed pegs, A, B, in a horizontal line, its extremities being free; find the catenary when the pressure on each peg is a given force, k.

Result. If $w =$ weight of chain per unit length, $AB = 2a$, $c =$ parameter of required catenary,

$$e^{\frac{2a}{c}} = \frac{k^2}{w^2 c^2} - 1$$

is the equation which determines c, and therefore the catenary.

In particular, if $w = 10$ pounds per foot length, $k = 500$ pounds' weight, $AB = 20$ feet, we find $c = 28{\cdot}87$, and whole length of chain $= 81{\cdot}64$.

23. If in the last the pressures on the pegs are not given but to be the least possible, find the catenary.

Result. $e^{\frac{2a}{c}} = \dfrac{c}{a-c}$ determines c, and gives approximately $\dfrac{a}{c} = 1 \cdot 1075$, and whole length of chain $= 2 \cdot 733 \times AB$.

24. A given length of uniform chain has its ends fixed to two pegs, A, B, in a horizontal line; find the distance between the pegs when the pressure on each is given.

Result. If $2l =$ length of chain, $w =$ weight of it per unit length, $wh =$ pressure on peg, $AB = 2a$,

$$2a = \sqrt{h^2 - l^2} \log_e \frac{h+l}{h-l}.$$

Thus, if $w = 2$ pounds' weight per foot, pressure $= \frac{1}{4}$ ton weight, length of chain $= 100$ feet, we find $AB = 49 \cdot 72$ feet.

25. In a uniform catenary of given length with extremities fixed at two points, A, B, in a horizontal line, determine AB so that each terminal tension shall be twice the tension at the lowest point.

Result. $AB = \dfrac{2l}{\sqrt{3}} \log_e (2 + \sqrt{3}) = \cdot 76 \times l$, where $2l =$ length of chain.

26. In the same case determine AB if the terminal is n times the horizontal tension.

Result. $AB = \dfrac{2l}{\sqrt{n^2 - 1}} \log_e (n + \sqrt{n^2 - 1})$.

27. If a uniform chain of given length, L, hangs with free extremities over two fixed supports, A, B, in a horizontal line, show that there will be two figures of equilibrium or none according as

$$L > \text{or} < e \, . \, AB,$$

where $e =$ Napierian base $= 2 \cdot 71828 \ldots$.

28. Show that for a given uniform catenary, AVB (V being the lowest point), hanging from two fixed extremities, A, B, in a horizontal line, the value of the parameter, c, may be thus obtained: at a distance from B equal to $\frac{1}{4}$ AB let a vertical line be drawn meeting the curve in P; then c is the third proportional to the semi-depth and the arc VP.

(The *depth* of the catenary is the perpendicular from V on AB.)

29. Show that if the arc VB of a catenary (V being the lowest point) is divided into any number of equal parts, and from the points of division ordinates be drawn to the directrix, the areas of all the strips thus formed are all equal.

30. A uniform chain of length $2l$, with free extremities, hangs over two fixed supports, A, B, in a horizontal line; show that if c is the

parameter of either of the two catenaries in which it may hang, the depth of the centre of gravity of the chain below AB is

$$\frac{l^2+c^2-2ac}{4l},$$

where the span $AB = 2a$. (Hence the catenary with the larger value of c is the stable, and the other the unstable, form.)

31. If in the last the span AB is 8 feet and the whole length of the chain 30 feet, prove that approximate values of the parameters of the two figures of equilibrium are 2 and 10 feet.

32. In the same case if $AB = 10$, and the whole length $= 60$, show that the parameters of the two figures are

$$1{\cdot}765 \text{ and } 24{\cdot}45,$$

and that the corresponding depths of the centre of gravity of the chain below AB are 7·398 and 10·418.

33. There are two equally rough horizontal tables of the same height separated by a distance, $2a$, which may be varied; a given uniform chain rests partly on each table, with equal lengths, f, on them and a given length, 2λ, hanging freely in the space between them; determine how far the tables can be drawn apart before the chain slips.

Result. If $\mu =$ coefficient of friction, the parameter, c, of the catenary when slipping is about to take place is given by the equation $c = \sqrt{\mu^2f^2-\lambda^2}$; and then $ce^{\frac{a}{c}} = \lambda+\mu f$ determines a.

Thus, if $\mu = \frac{1}{2}$, $f = 34$ feet, $\lambda = 15$; then $c = 8$, and $a = 11{\cdot}09$ feet.

34. A uniform chain 200 feet long and having a mass of 10 pounds per foot length has one end, A, fixed while the other end, B, is attached to a mass of 5200 pounds which rests on a rough horizontal table ($\mu = \frac{1}{2}$) close to its edge, the top of the table being at the same level as A; find the greatest value of the distance AB consistent with equilibrium.

Result. $AB = 194{\cdot}4$ feet.

35. If in Q. 33 one of the tables is higher than the other, find the greatest horizontal distance between them.

Result. If h is the difference of their heights, we have

$$16c^2\lambda^2 = (4\lambda^2-h^2)\{(2\mu f-h)^2-4\lambda^2\},$$

which determines the parameter of the catenary, and also

$$e^{\frac{D}{c}} = \frac{2\mu f-h+2\lambda}{2\mu f-h-2\lambda},$$

where D is the horizontal distance required.

In particular, if $2\lambda = 30$, $h = 4$, $f = 34$, $\mu = \frac{1}{2}$, the horizontal distance D is zero.

36. An endless chain 78 inches long hangs in two festoons over two fixed supports, A, B, in a horizontal line 26 inches apart; find the parameters of these festoons.

Result. They are 6·2 and 21·8, approximately, and of course the two festoons have the same directrix. There is, of course, another figure of equilibrium in which the two festoons coincide, making a single catenary.

37. If P, P' are any two points on a uniform catenary the arc-distances of which from the lowest point are s, s', show that

$$\sqrt{(s \pm s')^2 - (y - y')^2} = c\left(e^{\frac{x \pm x'}{2c}} - e^{-\frac{x \pm x'}{2c}}\right),$$

and hence show how to describe a catenary of given length through any two assigned points. (The problem is the same as that of the description of a catenary with a horizontal span.)

38. A uniform chain hangs with its extremities, A, B, fixed in a horizontal line; if a mass of given weight is suspended from the lowest point of the chain, find the altered figure.

Result. If the weight of the suspended mass = weight of a length $2p$ of the chain, $2l$ = length of chain, $AB = 2a$, we have

$$e^{\frac{a}{c}} = \frac{\sqrt{(l+p)^2 + c^2} + l + p}{\sqrt{p^2 + c^2} + p},$$

which determines the parameter of each of the equal catenaries formed.

Thus, if $AB = 20$, $2l = 50$, $2p = 40$, the value of c is approximately 13¼.

39. If the loaded chain in the last example, instead of having fixed ends, passes over the two fixed supports, A, B, and hangs with free ends, find the figures of equilibrium.

Result. With the same notation, we find

$$(l+p)\,e^{-\frac{a}{c}} = \sqrt{p^2 + c^2} + p,$$

to solve which for c, construct the curves whose equations are

$$y = (l+p)\,e^{-\frac{a}{x}},$$
$$y = \sqrt{p^2 + x^2} + p,$$

the first of which is drawn from a table of logarithms, and the second is a hyperbola with one vertex at the origin and the other on Oy at a distance p from O; this hyperbola is constructed by the well-known rule when we have a point and the asymptotes.

There are two points of intersection or none, and therefore two solutions or none.

Thus, if the whole length of the chain = 120, $AB = 20$, $2p = 40$, the values of c are 19·6 and 34·6, nearly. If the length of the chain were 100, while the other data remain the same, the curves do not intersect, and there is no equilibrium possible.

40. A uniform chain is suspended from two fixed points, A, B, in a horizontal line, and it is partially immersed in a liquid; find the figure of equilibrium.

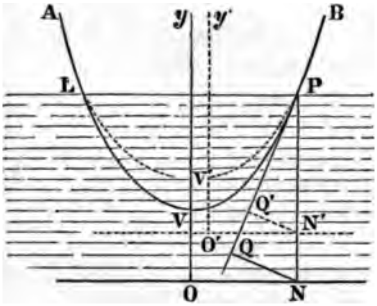


Fig. 8.

If the chain enters the liquid at P, the portion, BP, in the air and that, PVL, in the liquid belong to two distinct catenaries which have, however, a common tangent at P, their curvatures being different.

The air portion, BP, if continued below the liquid would form a catenary whose lowest point, V', is higher than V, the lowest point of the catenary PVL.

Let w = weight of chain per unit length, λ = weight of equal volume of liquid, h = height of B above LP, $2a = AB$; let the directrices of the two catenaries be ON, $O'N'$; their parameters c, c'; the ordinate $PN = y$, $PN' = y'$; arc $VP = s$, $V'P = s'$; then

$$(w-\lambda)y = wy',$$
$$(w-\lambda)c = wc',$$
$$(w-\lambda)s = ws',$$
$$(w-\lambda)\rho = w\rho',$$

where ρ and ρ' are the radii of curvature of the two curves at P. Also if x, x' are the distances of P from the verticals Vy, $V'y'$,

$$(w-\lambda)x = wx'.$$

Expressing the fact that $VP + PB = l$ = half length of chain, and putting $k = \frac{w-\lambda}{w}h$, we have

$$\frac{\lambda}{w}\sqrt{y^2-c^2} + \frac{w-\lambda}{w}\sqrt{(y+k)^2-c^2} = l. \quad \ldots\ldots\ldots\ldots (1)$$

We can also show that

$$\frac{\lambda}{w}\log_e\frac{y+\sqrt{y^2-c^2}}{c} + \frac{w-\lambda}{w}\log_e\frac{y+k+\sqrt{(y+k)^2-c^2}}{c} = \frac{a}{c}; \ldots (2)$$

and from these equations the two unknowns y and c are to be found.

In particular, if $AB = 100$, $h = 40$, $\lambda = \frac{1}{2}w$, $2l = 300$ = length of chain, we find that $c = 18$ approximately, and $PN = y = 23$, from which the dimensions of both catenaries are easily found.

[This and several other questions relating to catenaries have been discussed by the author in two papers in *Engineering*, vol. lxi, pp. 173, 303.]

41. A catenary of uniform strength is to be made with a mass of material equal to 400 pounds, and with a span of 36 feet; the density of the material is 100 pounds per cubic foot, and there is to be an intensity of stress equal to 20 pounds' weight per square inch in the catenary; find the curve.

Result. The depth of the curve is 6·03 feet, the area of its cross-section at the lowest point is about 13·9 square inches, and at the highest 17·1, the tangent at this point making an angle of 54° 11′ with the vertical.

NOTE.—The value of T in example 12, p. 252, is incorrect. The correct result is

$$T = \frac{P}{4}(\operatorname{cosec} \alpha - 2 \operatorname{cosec} \beta + 4 \sin \beta),$$

or

$$T = -P \cdot \frac{\sin(\alpha + 2\beta) - \sin \beta}{2 \sin \alpha \sin \beta},$$

so that T may be positive or negative, and will be zero when the polygon is regular. The angles α and β are connected by the relation

$$4 \cos \alpha \cos \beta = 1.$$

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